A Theory of Endogenous Asset Fire Sales, Bank Runs, and Contagion*

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Abstract

In a global-games framework, we endogenize asset fire sales, bank runs, and contagion by emphasizing the lack of information: asset prices collapse because assets sold by illiquid banks cannot be distinguished from those by insolvent banks. However, it is the collapse of prices that leads to runs and contagion in the first place. We derive three policy implications. (1) Increasing capital can exacerbate fire sales because runs on well-capitalized banks signal high risks. (2) Asset purchase programs break down the vicious cycle and promote financial stability. (3) Regulatory disclosure involves a trade-off: while favorable disclosures mitigate fire sales, acknowledging crises aggravates financial vulnerability.

Keywords: Bank run, Global games, Asymmetric information, Capital, Asset purchase program, Regulatory disclosure

JEL Classification: G01, G11, G21

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1 Introduction

The recent banking crisis highlights the importance of liquidity risk. During the crisis, market liquidity evaporated, and asset prices dropped sharply. At the same time, as funding liquidity dried up, even well-capitalized banks found it difficult to roll over their short-term debt and had to resort to central banks.

Market liquidity and funding liquidity are closely related. It is widely acknowledged that market illiquidity contributes to funding illiquidity. As market liquidity diminishes, potential fire-sale losses from early liquidation make creditors panic. As a result, the creditors can stop rolling over their short-term debt and thus deprive a healthy financial institution of its funding. Such funding liquidity risk has been emphasized by, among others, Morris and Shin (2000), Rochet and Vives (2004) and Goldstein and Pauzner (2005). However, the literature has so far ignored the fact that funding illiquidity also feeds market illiquidity: bank runs can lead to fire sales, depress asset prices, and in extreme cases, freeze up markets.[1]

The current paper fills this gap by presenting a model where asset fire sales and bank runs mutually reinforce each other. The feedback is driven by the lack of information. That is, a bank that fails because of illiquidity cannot be distinguished from one that fails because of fundamental insolvency.[2] We incorporate such information friction into a global-games-based bank run model and endogenize fire-sale prices: asset prices are determined by the micro-foundation of information asymmetry. As a defining feature that distinguishes the current model from the literature, we have market participants' beliefs, asset prices, and bank runs, all endogenous and jointly determined in a rational expectations equilibrium. We prove that an equilibrium exists and is unique. We also extend the theoretical framework to model financial contagion. We show that contagion can be driven by creditors' beliefs and emerge as a result of multiple equilibria.

The intuition of our model is as follows. When asset buyers cannot distinguish assets sold by an illiquid bank from those by an insolvent one, their offered price will be distorted downwards. As a result, the illiquid bank cannot recoup a fair value for its assets on sale. This friction leads to a vicious circle. To begin with, low asset prices fuel bank runs: expecting fire-sale losses caused by other creditors' early withdrawals, a creditor has incentives to join

[1] For example, Acharya and Roubini (2009) documented how the early liquidation of two highly levered Bear Stearns-managed hedge funds stressed the price of asset-backed securities.

[2] This is wildly recognized in the literature as well as in practice. In fact, it is considered the main challenge for central banks to perform as a lender of last resort. See, for example, Freixas, Rochet, and Parigi (2004).

the run. However, it is the run and early liquidation, by pooling illiquid banks with insolvent ones, that leads to the drop in asset price in the first place. In this sense, creditors' pessimistic expectation realizes itself. With one more ingredient—an aggregate state, we show that financial contagion can happen in a similar manner. As the uninformed asset buyers form rational expectations, they revise their beliefs about the aggregate state downwards upon observing any bank runs. Their deteriorating expectations lower the asset price that they are willing to pay, which, in turn, precipitates contagious runs on other banks.

Our model delivers three policy insights. First, while our paper confirms that well-capitalized banks have larger buffers against fire-sale losses, our analysis also reveals that once asset prices are endogenous, increasing capital can also have unintended consequences for liquidity via buyers' beliefs. In particular, buyers' posterior beliefs about a bank's asset value will deteriorate when a run happens to the bank, and the deterioration is particularly strong when the bank maintains a high capital ratio. Because well-capitalized banks can sustain large losses, if a run happens to such a bank, the bank's losses must be unusually high. Therefore, given that a bank faces a run, buyers' valuation of its assets decreases in its capital level. Their low willingness to pay makes the run more likely to happen in the first place. In its extremity, the model predicts that increasing capital does not reduce bank funding liquidity risk at all.

Second, our model confirms the effectiveness of asset purchase programs in promoting financial stability. We show that even if regulators have no better information than other market participants, an asset purchase program can improve financial stability at no social cost. In an asset purchase program where a regulator buys bank assets at a committed price, the vicious cycle fueled by beliefs will break down. We emphasize that the lack of commitment by typical asset buyers can be at the very root of financial instability. In particular, typical asset buyers behave according to their rational beliefs, and would avoid losses in every perceived state. This would generate the aforementioned vicious cycle because buyers' pessimistic beliefs can lead to negative market outcomes (e.g., more bank runs) which in turn justify themselves. A regulator with commitment power, on the other hand, can resist such pessimistic belief updating. This allows him to promote financial stability while breaking even from an ex-ante perspective.

Finally, when regulators have better information than typical market participants, the model highlights a trade-off for regulatory disclosures.[3] Our paper considers such regulatory disclosures to be a double-edged sword: if the disclosed information reassures market participants,

[3]For instance, whether or not regulators should disclose information concerning their assistance programs.

banks can be saved from illiquidity. However, if the disclosure adds to pessimistic market beliefs, the disclosure itself can lead to financial vulnerability. This is because once the severity of the problem is acknowledged, market participants will further revise down the expected performance of all banks', leading to greater fire-sale losses and triggering illiquidity, even for healthy banks.

Our theoretical framework is related to the literature on bank runs and financial contagion. Since Diamond and Dybvig (1983), the literature has been concerned with the financial fragility caused by bank runs.[4] Following their seminal contribution, there was a debate as to whether bank runs are due to pure panic or unfavorable information on banks' fundamentals.[5] The gap between the panic and fundamental view was bridged by the application of global games. Using the concept, papers such as Morris and Shin (2000), Rochet and Vives (2004) and Goldstein and Pauzner (2005) refined the multiple equilibria in Diamond and Dybvig (1983) and emphasized the role of fire-sale losses in causing bank runs. That is, to prevent runs, an extra buffer of cash flow is needed against fire-sale losses. A bank that fails to provide the extra buffer become "solvent but illiquid"—being able to repay its debts in full if no run happens, but to be liquidated early if its creditors do not roll over their debt. But a limitation of the existing models is that they build on the simplifying assumption of exogenous fire-sale losses,[6] so that the models ignore the explicitly exclude the possibility for bank runs to affect asset prices. In contrast, the current paper explores the relationship: as it is difficult to distinguish illiquid banks from insolvent ones, the adverse selection causes low asset prices and fire-sale losses.[7]

A natural corollary of assuming an exogenous fire-sale price is that funding liquidity risk will be always reduced by higher capital, because the returns generated on capital add to the buffer against fire-sale losses. With endogenous fire-sale prices the current paper takes a broader view: while acknowledging the buffer effect of capital, we point out that greater capital can also contribute to illiquidity via buyers' pessimistic inference.

[4]It should be mentioned that some papers also consider the positive role of bank run as disciplinary device: Calomiris and Kahn (1991) and Diamond and Rajan (2001).

[5]Papers emphasizing banks' weak fundamentals in causing runs include Chari and Jagannathan (1988), Jacklin and Bhattacharya (1988) and Allen and Gale (1998). Friedman and Schwartz (1963) provide empirical support for the panic view. Contradicting evidence in favor of the fundamental view is present in Gorton (1988), Calomiris and Gorton (1991) and Calomiris and Mason (2003).

[6]For example, Rochet and Vives (2004) assume an exogenous fire-sale discount and Morris and Shin (2009) assume an exogenous haircut.

[7]While the current paper justifies the low asset price by informational frictions, low asset prices can also be explained by fixed short-term cash supply—the cash-in-the-market argument pioneered by Shleifer and Vishny (1992) and Allen and Gale (1994).

In predicting an interaction between market liquidity and funding liquidity, our model is most closely related to Brunnermeier and Pedersen (2009), who emphasize a haircut constraint on a speculator that supplies liquidity to a financial market with limited participation. In their model, asset prices are volatile because there is an asynchronization between selling and buying. This paper differs from theirs in two aspects. First, the funding liquidity risk arises as a result of equilibrium bank runs caused by the wholesale creditors' coordination failures. Second, this paper emphasizes the asymmetric information on asset qualities, and how such adverse selection causes asset illiquidity.

In our paper, contagion is generated not only by the actual realization of a common risk factor but also by its perception: A bank failure casts shadow on the perceived common risk factor; and the resulting negative informational externalities affect all the other banks. This observation is mostly related to the literature of information contagion, as exemplified by Acharya and Thakor (2011) and Oh (2012).[8] Compared to the existing work, the current paper emphasizes the self-fulfilling nature of such contagion and the two-way feedback between runs and asset fire sales.

On the application to capital requirements, the paper relates to a few papers that show increased capital requirements can increase bank risk. Martinez-Miera (2009) argues that equity increases banks' cost of funding, which leads to higher loan rates and spurs risk-taking by borrowers. As a result, banks' portfolio risk rises passively. Hakenes and Schnabel (2007) argue that a higher capital requirement erodes charter value and induces banks to actively take high risks; when the higher capital requirement decreases credit supply, it also leads to borrower risk-taking via a hike in loan rate. What all these papers have in common is that they all focus on solvency risk. To the best of our knowledge, the current study is the first to show capital can contribute to illiquidity, contagion and systemic risk.

On asset purchase programs, our paper is mostly related to the empirical policy evaluation in Veronesi and Zingales (2010). The authors used Black-Scholes-Merton model to evaluate the ex-ante costs and benefits of "Paulson's Plan", and concluded that the intervention yielded "a net benefit between \$86 and \$109 bn". The policy evaluation in our theoretical model also takes an ex-ante perspective, and confirms the effectiveness of asset purchase programs in promoting financial stability.

[8]There are other approaches to model contagion. For instance, Freixas and Parigi (1998) and Freixas, Parigi, and Rochet (2000) model the direct linkages of banks through payment system. Allen and Gale (2000) emphasize the role of interbank market. Gorton (1988) studies banks' common risk exposures directly contribute to systemic risk.

The discussion on disclosure policy is most related to several recent papers on the instability consequences of public signals: Morrison and White (2013) concern that a public bailout can reveal regulatory deficiency and make market participants lose their confidence in all other banks under the same regulation. Dang, Gorton, and Holmström (2010) show that a public signal could increase adverse selection to debt-like securities that are otherwise information insensitive. Wang (2013) empirically documents that after individual banks were identified in Trouble Asset Relief Program (TARP), bank run probabilities, as reflected in CDS spread and stock market abnormal returns, rose dramatically, an outcome the author attributes to the bad news nature of public bailout. Our paper abstracts from specific policy announcements and shows that as long as market participants believe the regulator is better informed, any regulatory action and announcement concerning banks' common risk exposure may generate financial contagion.

The paper proceeds as follows. Section 2 lays out the model. Section 3 presents the baseline bank-run model with endogenous fire-sale prices. With only one bank and one state, the baseline model allows us to discuss the first policy issue such as whether higher capital can effectively reduce funding liquidity risk. In Section 4, we analyze contagion in a fully-fledged model with two banks and two states. We show that even if a regulator has no better information than typical market participants, asset purchase programs can improve financial stability at no social cost. In Section 5, we discuss the trade-off associated with regulatory disclosures. Section 6 concludes.

2 Model setup

We consider a three-date ($t = 0, 1, 2$) economy with two banks.[9] At $t = 0$, banks are identical. Each of them holds a unit portfolio of long-term assets, and finances them with equity E, retail deposits F, and short-term wholesale debts $1 - E - F$. There are two groups of active players: banks' wholesale creditors and uninformed buyers of banks' assets. Both groups of players are risk neutral. We assume that retail deposits are fully insured so that depositors act only passively. Since their claims are risk free, the depositors will always hold their claims to maturity, and demand only a gross risk-free rate which we normalize to 1. We also assume that the financial safety net is provided to banks free of charge. We consider banks

[9] It should be emphasized that all results of the current paper can be generalized to a N-bank case.

as contractual arrangements among claim holders, designed to fulfil the function of liquidity and maturity transformation (Diamond and Dybvig (1983)). Therefore, banks in our model are passive, with given loan portfolios and liability structures.

Banks' wholesale debt is risky, demandable, and raised from a continuum of creditors. Provided that a bank does not fail, a wholesale debt contract promises a gross interest rate $r_D > 1$ at $t = 2$, and qr_D if a wholesale creditor withdraws early at $t = 1$. Here $q < 1$ reflects the penalty for the early withdrawal. A bank run occurs if a positive mass of wholesale creditors withdraw funds from their bank at $t = 1$. For the ease of future exposition, we denote by D_1 the total amount of debt a bank needs to repay at $t = 1$ if *all* wholesale creditors withdraw early, and by D_2 the total amount of debt a bank needs to repay at $t = 2$ if *no* wholesale creditor withdraws early.

$$D_1 \equiv (1 - E - F)qr_D$$
$$D_2 \equiv (1 - E - F)r_D + F$$

A bank's portfolio generates a random cash flow $\tilde{\theta}$ at $t = 2$. For simplicity, we assume that $\tilde{\theta}$ follows a uniform distribution on $\left[\underline{\theta}_s, \overline{\theta}\right]$, and the random cash flows of the two banks are independent and identically distributed. Subscript s denotes the realization of an aggregate state that affects both banks. There are two possible states, G and B (e.g., housing market boom or bust), and the two states occur with equal probability. With $\underline{\theta}_G > \underline{\theta}_B$, State G is more favorable than State B. Therefore, the value of a bank's assets is not only affected by its idiosyncratic risk (the realization of $\tilde{\theta}$) but also by the aggregate risk s. On the other hand, $\overline{\theta}$ is assumed to be the same across states. This reflects the fact that banks hold mostly debt claims whose highest payoffs are capped by their face values. We make the following three further assumptions on parameters.

$$D_2 > \underline{\theta}_s \tag{1}$$
$$\left(\underline{\theta}_B + \overline{\theta}\right)/2 > D_2 \tag{2}$$
$$F > D_1 \tag{3}$$

As D_2 denotes a bank's total debt obligation at $t = 2$, inequality (1) states that there is a positive probability of bankruptcy in each state. Inequality (2) states that, in the absence of bankruptcy costs, even if the realization of the state is unfavorable, the expected cash flow from a bank's assets is still greater than its debt obligations, so that bank lending is viable. Finally, inequality (3) states that a bank's retail debts exceed its wholesale debts, which is a realistic scenario and helps to simplify the analysis of bank run games.[10] We also assume that bankruptcy costs are sufficiently high such that once a bank fails, the wholesale creditors get zero payoff and only a deposit insurance company obtains the residual value.[11]

Banks' assets are long-term, taking two periods to mature. In particular, we assume that at $t = 1$ the assets cannot be physically liquidated. Therefore, if a wholesale run happens, to meet the liquidity demand, a bank has to financially liquidate its assets in a secondary asset market, and sell them to outside asset buyers. Because early liquidation is costly in this model, a bank will sell its assets if and only if it faces a bank run.

2.1 Secondary asset market

Potential buyers in the secondary asset market are uninformed: they are unable to observe either the aggregate state s or any bank's cash flow θ. Yet, they can observe the number of bank runs, and based on the observable outcome, form rational expectations about the quality of assets on sale. In this two-bank setup, there are three distinct outcomes from the buyers' perspective, i.e., the number of bank runs $N = 0$, 1, or 2.

We assume the following sequential moves between asset buyers and wholesale creditors.[12] Asset buyers first post a price scheme $\mathbb{P} = (P_1 \;\; P_2)$, and offer to purchase bank assets on sale at price P_1 when the number of bank runs $N = 1$, and P_2 when $N = 2$. Having observed the price scheme, wholesale creditors play a bank run game, making their individual decisions simultaneously on whether to withdraw their funds early. In case any bank run happens, transactions take place at the offered price, and assets are transferred to buyers.

The price scheme $\mathbb{P}$ is complete in the sense that an asset price is specified for each distinct outcome of the bank run game where bank assets are on sale. Depending on the number of runs observed, the prices that buyers offer can differ. In fact, in the absence of commitment power, the asset buyers' decisions need to be time consistent so that they will not revoke their posted

[10]The condition is more than a technical assumption. It is realistic in the sense that despite of the rapid growth of wholesale funding, most commercial banks and bank holding companies are still financed more by retail deposits than wholesale debt.

[11]It should be noted that this is off equilibrium in the model, because the wholesale debt is demandable and the wholesale creditors will withdraw early at $t = 1$, before the bank fails.

[12]It should be emphasized that the results of the model are robust to timing. If asset buyers and wholesale creditors move simultaneously, one can derive the same results based on the rational expectations of both parties.

price after the outcomes of bank run games are revealed. As a result, the price P_1 and P_2 will have to reflect buyers' posterior beliefs on asset qualities. As buyers form different posterior beliefs when observing different numbers of bank runs, their offered prices will vary with the number of bank runs.

The asset market is assumed to be perfectly competitive, and the buyers compete in the price schemes that they offer. In equilibrium, based on their posterior beliefs, the asset buyers should perceive themselves breaking even in expectation when purchasing bank assets at their posted prices. As the buyers make time-consistent decisions and do not revoke their offers, they must make no loss for any realized number of bank runs.

2.2 Bank run game

The demandable nature of wholesale debt allows creditors to withdraw their funds before a bank's assets mature, forcing the bank to liquidate its assets prematurely. When assets are sold for less than their fundamental values, there will be an early liquidation loss, or an asset fire sale. While the creditors who withdraw early can avoid suffering from the fire sale, those who do not withdraw will receive zero payoffs if the bank fails. As a result, creditors' actions to withdraw display strategic complementarities, and it can be in the interest of all creditors to run on a bank that is otherwise solvent.

A bank run game of complete information can have two strict equilibria: all creditors withdraw from the bank, and nobody withdraws. To refine the equilibria, we take the global-games approach pioneered by Carlsson and Van Damme (1993) and study games with incomplete information, where common knowledge on θ does not exist among creditors. We assume that at the beginning of $t = 1$, both aggregate risk (State s) and idiosyncratic risk (cash flow θ) have been realized, but the information is not fully revealed to players. For a given bank, each individual creditor privately observes only a noisy signal $x_i = \theta + \epsilon_i$. The noise ϵ_i is drawn from a uniform distribution with a support $[-\epsilon, \epsilon]$, where ϵ can be arbitrarily small. Based on their private signals, the creditors play a bank-run game with each other. Each of the creditors has two possible actions: to wait until maturity or to withdraw early, and follows a threshold strategy: withdraw early if and only if their individual private signal is lower than a critical level $\hat{x}$. In this two-bank setup, we also assume that each creditor holds claims in both banks, and observes independent noisy signals for both banks' cash flows.[13]

The maturity mismatch between banks' liabilities and assets, together with potential asset fire sales, exposes banks to the risk of runs. In particular, a run and premature liquidation at $t = 1$ can cause the failure of a bank that is otherwise solvent at $t = 2$. In order to reassure its creditors not to withdraw early, a bank has to be more than merely solvent, and should be able to absorb potential fire-sale losses. This implies a critical cash flow $\hat{\theta} > D_2$ for a bank to survive a run. The distance between $\hat{\theta}$ and D_2 provides a measure of financial vulnerability. Moreover, a lower asset price implies greater fire-sale losses, and a higher critical cash flow $\hat{\theta}$ for a bank to survive a run.

Given our assumption that bankruptcy costs are sufficiently high, if a bank is to fail at $t = 1$, a wholesale creditor will receive zero payoff whether he withdraws early or not. In this case of indifference, we assume that the creditor will always withdraw. One justification is that wholesale creditors receive reputational benefits by running on a bank that is doomed to fail.[14]

2.3 Asymmetric information on cash flow θ

We assume that asset buyers can solve the creditors' bank run game and thereby form rational beliefs on the qualities of assets on sale. In particular, they know that a bank will be forced into an asset sale if and only if its cash flow is below $\hat{\theta}$. However, the lack of more detailed information makes solvent banks (those with $D_2 \leq \theta < \hat{\theta}$) indistinguishable from the insolvent ones (those with $\theta < D_2$). An equilibrium asset price reflects only the average quality of assets on sale, so a bank with cash flow θ greater than the price but less than $\hat{\theta}$ will face an asset fire sale.

A lower asset price pushes $\hat{\theta}$ the critical cash flow necessary to avoid a run upwards. Thus, there will be two-way feedback between asset fire sales and bank runs. When asset buyers offer a low price for a bank's assets, a run is triggered, which generates the pooling of assets, and thus fully justifies the low asset price offered in the first place. As a result, both fire sales and bank runs can reinforce each other.

[13] It is not uncommon for institutional investors to hold demandable debt claims in multiple banks. A similar assumption can be found in Goldstein and Pauzner (2004).

[14] The reputational benefit may come from the fact that the creditor makes a "right decision". For more detailed discussion on this assumption, see Rochet and Vives (2004). The authors argue that the vast majority of wholesale deposits are held by collective investment funds, whose managers are compensated if they build a good reputation, and penalized otherwise.

2.4 Belief updating on State s

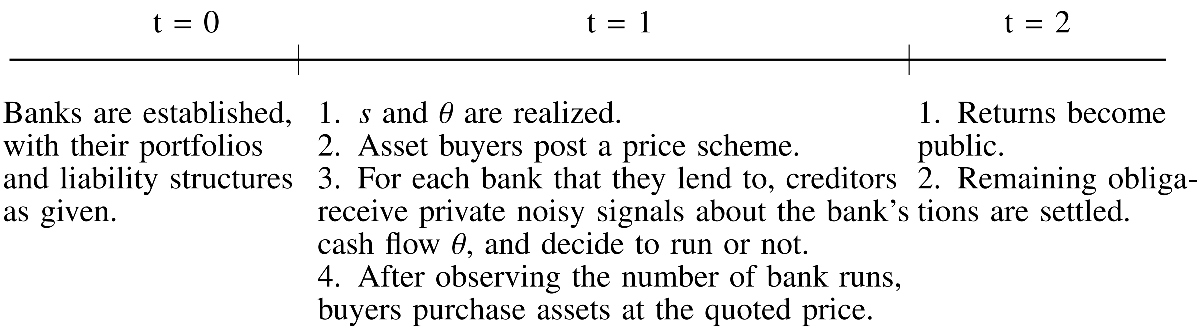
While asset buyers hold a prior belief that State B and G occur with an equal probability, after observing any bank runs, they update their beliefs according to Bayes' rule and consider State B to be more likely. The pessimistic belief updating can lead to financial contagion. In particular, a bank may face no runs if the other bank does not face a run, but will face a run if the other one does. This defines financial contagion in our model.

In the current model, financial contagion is self-fulfilling too. When observing more bank runs, asset buyers infer State B to be increasingly likely and reduce their offered asset prices accordingly. The fear of increased liquidation losses makes wholesale creditors panic even more, and leads to simultaneous bank runs in the first place.

2.5 Timing

The timing of the model is summarized in Figure 2. Events at $t = 1$ take place sequentially.

Figure 1: Timing of the game



3 Self-fulfilling bank runs and fire sales

Depending on the realization of $\tilde{\theta}$, the model can have two types of equilibria: one type with bank runs, and the other without. A market equilibrium with bank runs consists of two components. First, the bank run games feature threshold equilibria. That is, when N runs happen and bank assets are sold for an equilibrium price P_N^*, a bank will experience a run if and only if its cash flow is lower than a unique threshold $\theta_N^* \equiv \hat{\theta}(P_N^*)$, $N \in \{1, 2\}$. Second, the competitive asset market is in a rational expectations equilibrium. That is, asset buyers form a rational belief about the quality of assets on sale based on the observed number of bank runs N. In particular, they anticipate $\theta < \hat{\theta}(P_N^*)$, and Bayesian update their beliefs on State s. According to such posterior beliefs, asset buyers who purchase bank assets at an equilibrium price P_N^* should perceive themselves to break even in expectation. Moreover, the buyers should find themselves unable to profitably deviate from bidding P_N^*.

Definition. *Denote $\hat{\theta}(P_N)$ the threshold equilibrium of the bank run game for a given asset price P_N; and $P_N\left(\hat{\theta}\right)$ the price scheme by which asset buyers break even in expectation for a given threshold $\hat{\theta}$ and their rational beliefs about θ and s. An equilibrium associated with N bank runs, $N \in \{1,2\}$, is defined by an equilibrium critical cash flow $\theta_N^* \equiv \hat{\theta}\left(P_N^*\right)$ and an equilibrium asset price $P_N^* \equiv P_N\left(\theta_N^*\right)$. The combination of θ_N^* and P_N^* is such that: for the N bank runs in the economy, (1) a successful run happens to a bank if and only if the bank's cash flow is lower than θ_N^*; (2) the competitive asset market is in a rational expectations equilibrium: asset buyers form rational beliefs about State s and the quality of assets on sale, and based on their posterior beliefs, perceive themselves to make zero profit in expectation by purchasing bank assets at P_N^*. Furthermore, the buyers cannot make any profitable deviation.*

It takes four steps to obtain the equilibrium.

- First, we show that equilibrium asset prices P_N^* cannot be lower than D_1 or higher than D_2 (subsection 3.1). This restricts the set of candidate equilibria and will facilitate the solution of bank run games.
- Second, solving the model using backward induction, we start with creditors who move last and solve the bank run game using the concept of global games. For a *given* asset price $P_N \in (D_1, D_2)$, we derive a unique critical cash flow $\hat{\theta}(P_N)$, so that a bank run will happen if and only if the bank's cash flow $\theta < \hat{\theta}(P_N)$ (subsection 3.2).
- Third, we characterize asset buyers' posterior beliefs on asset qualities when N bank runs occur. In particular, they expect only those assets with quality $\theta < \hat{\theta}(P_N)$ to be on sale, and update their beliefs about State s using Bayes' rule. It should be emphasised that the buyers' rational beliefs are functions of asset prices that they offer (subsection 3.3).
- Finally, we solve for the equilibrium of the model by examining equilibrium asset price schemes. As asset buyers offer different prices given different numbers of bank runs, we solve for equilibrium prices P_N^* for each $N \in \{1,2\}$. For N observed bank runs, in a

competitive equilibrium, P_N^* should be equal to the expected asset quality based buyers' posterior beliefs (subsection 3.3).

To illustrate the main intuition behind the feedback between bank runs and fire sales, we present in subsection 3.4 a simplified version of the model where there is only one state so that asset buyers cannot update their beliefs on State s. This simplification allows us to derive a closed-form solution to our model, and is sufficient to generate some interesting results such as the unintended liquidity consequences of bank capital. The fully-fledged model with different states and asset buyers' belief updating on s is analyzed in section 4.

3.1 Restricting the set of candidate equilibria

Since an equilibrium price cannot be negative, a candidate equilibrium price P_N^* can only fall into one of three regions, $0 \leq P_N^* \leq D_1$, $D_1 < P_N^* < D_2$, and $P_N^* \geq D_2$. We discuss the existence and uniqueness of equilibrium for each of the three regions, and show that any equilibrium price P_N^* cannot be lower than D_1, nor greater than D_2 provided that $F > D_1$.

Suppose $P_N^* \geq D_2$. Then, for any bank with $\theta \in [D_2, \overline{\theta}]$, it is suboptimal for its wholesale creditors to withdraw early. This is because with $P_N^* \geq D_2$, an asset sale at $t = 1$ will not hurt the bank's capability to repay its liabilities at either $t = 1$ or $t = 2$. As a result, by running on the bank, a creditor will only incur the penalty for early withdrawal. This implies that whenever a run happens, it must be the case that the bank is fundamentally insolvent with $\theta < D_2$. Therefore, the highest asset quality that buyers can expect is D_2, with the expected quality strictly lower than that. Because asset buyers break even and pay a price equal to the expected quality, the price that the buyers are willing to pay must be strictly smaller D_2. This contradicts the presumption $P_N^* \geq D_2$.

Now, suppose $P_N^* \leq D_1$. Then, a bank with $\theta \in [\underline{\theta}_s, D_2]$ will fail for sure, either because sufficiently many creditors run at $t = 1$, or because of fundamental insolvency at $t = 2$. Under the assumption that wholesale creditors run on banks that are doomed to fail, we know that successful runs must happen to those banks with $\theta \in [\underline{\theta}_s, D_2]$. This implies that the expected quality of assets on sale is at least $(\underline{\theta}_B + D_2)/2$. As asset buyers only break even in equilibrium, the price they offer must be greater than that. Therefore, we have $P_N^* > (\underline{\theta}_B + D_2)/2 > D_2/2$. By the definitions of D_1 and D_2, we further have $D_2/2 = [(1 - E - F)r_D + F]/2 > [(1 - E - F)qr_D + F]/2 = (D_1 + F)/2$, which is in turn greater than D_1, provided $F > D_1$. Again, this contradicts the presumption $P_N^* \leq D_1$. We summarize these results in Lemma 1.

Lemma 1. *An equilibrium asset price cannot be less than or equal to D_2. And an equilibrium asset price cannot be greater than or equal to D_1 either, provided $F > D_1$.*

3.2 Threshold equilibrium for bank run games

We solve the model by backward induction, and start with the subgame of bank runs. We show that for a *given* price $P_N \in (D_1, D_2)$ the bank run game has a unique threshold equilibrium characterized by a critical cash flow $\hat{\theta}(P_N)$. A successful bank run happens if and only if the bank's cash flow is lower than $\hat{\theta}(P_N)$.

To solve for the optimal strategy of creditors, we first derive their payoffs for action "wait" and "withdraw" as functions of the number of other creditors who withdraw from the bank. Denote by $L \in [0, 1]$ the fraction of creditors who withdraw from the bank at $t = 1$. A bank that faces a total withdrawal of LD_1 can meet the demand for liquidity with a partial liquidation by selling a fraction f of its assets.[15]

$$f = \frac{LD_1}{P_N} < 1 \tag{4}$$

After liquidating a fraction f of its assets, the bank will fail at $t = 2$ if and only if the value of its remaining assets $(1 - f)\theta$ is lower than its remaining liabilities $F + (1 - L)(1 - E - F)r_D$. That is,

$$(1 - f)\theta \leq F + (1 - L)(1 - E - F)r_D. \tag{5}$$

Thus, a bank will fail at $t = 2$ if and only if the fraction of creditors' withdrawal exceeds a threshold L^c.

$$L \geq \frac{P_N[\theta - F - (1 - E - F)r_D)]}{(q\theta - P_N)(1 - E - F)r_D} = \frac{P_N(\theta - D_2)}{[\theta - P_N/q]D_1} \equiv L^c. \tag{6}$$

Such a $t = 2$ failure happens because the partial early liquidation incurs a cost of fire sale. When a sufficiently large number of creditors withdraw and the bank is forced to liquidate a significant share of assets prematurely, the remaining assets will not generate sufficient cash flows to meet the remaining liabilities. The creditors who withdraw early at $t = 1$ therefore can impose negative externalities on creditors who choose to wait.

Depending on the amount of early withdrawals L, a creditor's payoffs of playing withdraw or stay are tabulated as follows.

[15] Here $f < 1$ is guaranteed by $P_N > D_1$ and $L \leq 1$. Note that three factors contribute to a high fraction of asset liquidation: (i) a large number of early withdrawals, (ii) low market price P for assets on sale, and (iii) a high level of wholesale debt.

	$L \in [0, L^c)$	$L \in [L^c, 1]$
withdraw	qr_D	qr_D
stay	r_D	0

Note that if a creditor withdraws, his payoff will always be $W_{run}(L) = qr_D$. Instead, if he waits, his payoff depends on the action of other creditors.

$$W_{wait}(L) = \begin{cases} r_D & L \in [0, L^c] \\ 0 & L \in [L^c, 1] \end{cases}$$

Defining the difference between the creditor's payoffs of withdraw and stay as $DW(L) \equiv W_{run}(L) - W_{wait}(L)$, one has

$$DW(L) = \begin{cases} -(1-q)r_D & L \in [0, L^c] \\ qr_D & L \in [L^c, 1] \end{cases}$$

The strategic complementarity is clear: when a sufficiently large number of other creditors choose to withdraw ($L > L^C$), a wholesale creditor receives better payoff from withdrawing than from waiting. In fact, when there is complete information on θ, the bank run game has two equilibria in which either all creditors withdraw or all creditors wait. We refine the multiple equilibria using the technique of global games.

The analysis follows a standard global games approach. We give here the outline of the proof, and interested readers can refer to Appendix A for full details. First, we establish the existence of a lower dominance region $[\underline{\theta}_s, \theta^L]$, where it is a dominant strategy for all wholesale creditors to withdraw early, independently of the private signal that they receive. Similarly, we show there exists an upper dominance region $[\theta^U(P_N), \overline{\theta}]$, where it is a dominant strategy for all creditors to wait.[16] For the intermediate range $\theta^L < \theta < \theta^U(P_N)$, a creditor's payoff depends on the actions of other creditors. So, as a second step, we characterize a creditor i's ex-post belief about the other creditors' actions, conditional on his private signal $x_i = \theta + \epsilon_i$. The belief is a conditional distribution of L. The creditor then chooses his optimal action based on the ex-post belief and payoff function $DW(L)$. Finally, for the limiting case where the noise of the signal approaches zero, we obtain a unique threshold

$$\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N} \tag{7}$$

[16]One can derive the upper and lower bounds explicitly and show that $\theta^L = D_2$ and $\theta^U(P_N) = \frac{F}{1-D_1/P_N}$.

such that a successful bank run will happen if and only if the bank's cash flow $\theta < \hat{\theta}(P_N)$. The results are summarized in Proposition 1.

Proposition 1. *For a secondary market asset price $P_N \in (D_1, D_2)$, the bank run game has a unique threshold equilibrium: a successful run occurs to a bank if the bank's cash flow falls below a critical level* $\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N}$.

Proof. See Appendix A. □

Expression (7) establishes a one-to-one correspondence between the asset price P_N and the critical cash flow $\hat{\theta}(P_N)$. Note that the critical cash flow $\hat{\theta}(P_N)$ is decreasing in P_N. A lower asset price makes successful bank runs more likely.

3.3 Asset market equilibrium

The uninformed asset buyers observe neither θ nor State s, but they can form rational beliefs about the quality of asset on sale. First of all, they anticipate the threshold equilibrium for the bank run game to be characterized by $\hat{\theta}(P_N)$. Therefore, when N bank runs happen, the asset buyers form a rational belief that only those assets of quality $\theta < \hat{\theta}(P_N)$ will be on sale. Second, the asset buyers also update their beliefs about State s using Bayes' rule. We denote by $\omega_N^G\left(\hat{\theta}(P_N)\right)$ the buyers' posterior belief that $s = G$ when the observed number of bank runs equals N, and by $\omega_N^B\left(\hat{\theta}(P_N)\right)$ the posterior belief for $s = B$. It should be emphasized that the posterior beliefs depend on buyers' offered price P_N.

Note that two factors can contribute to asset fire sales. First, conditional on a bank run having happened, the cash flow of the bank must be lower than $\hat{\theta}(P_N)$. The buyers face an adversely selected asset pool in the sense that only those banks with low cash flow will be forced into asset sales. Second, any observed bank runs also indicate that $s = B$ is more likely. This further reduces the expected quality of assets on sale, which in turn reduces buyers' willingness to pay.

When the asset market is perfectly competitive, an equilibrium asset price must satisfy two conditions. First, based on their rational expectations about θ and s, the buyers should make zero expected profit by purchasing bank assets at the posted price. In other words, when there are N bank runs, an equilibrium asset price P_N^* equals the expected asset quality.

$$P_N^* = E\left[\theta | \theta < \hat{\theta}(P_N^*)\right] = \omega_N^G\left(\hat{\theta}(P_N^*)\right)\frac{\theta_G + \hat{\theta}(P_N^*)}{2} + \omega_N^B\left(\hat{\theta}(P_N^*)\right)\frac{\theta_B + \hat{\theta}(P_N^*)}{2} \tag{8}$$

Second, a buyer should not be able to make any profitable deviation by unilaterally bidding a higher price. Therefore, their expected net payoff, $E\left[\theta|\theta < \hat{\theta}(P_N), N\right] - P_N$, should not increase in P_N.

The equilibrium has a fixed-point representation: P_N^* should be a fixed point for function $E\left[\theta|\theta < \hat{\theta}(P_N), N\right]$. We show that for *each* $N \in \{1, 2\}$, the fixed-point equilibrium exists and is unique. We also verify that the equilibrium is stable in the sense that a buyer cannot profitably deviate by unilaterally bidding a higher price.

3.4 A baseline model

The feedback between a bank run and an asset fire sale can be examined without different aggregate states. Therefore, to illustrate the main intuition, we analyze a baseline case of our model with $\underline{\theta}_B = \underline{\theta}_G = \underline{\theta}$. In this case, buyers do not update their beliefs about State s, so their posted price scheme will consist of only one unified price P. For this baseline model, we denote market equilibrium by $\{\theta_e, P_e\}$, and obtain closed-form solutions.

As discussed, intelligent asset buyers can solve the subgame of bank runs and anticipate only those assets of quality $\theta < \hat{\theta}(P)$ to be on sale. On the other hand, when the asset market is in a competitive equilibrium, asset buyers who purchase banks' assets at the posted price should break even in expectation. Given their belief $\theta \sim U\left(\underline{\theta}, \hat{\theta}(P)\right)$, a candidate equilibrium price P_e must satisfy the following zero-profit condition:

$$P_e = \frac{\hat{\theta}(P_e) + \underline{\theta}}{2}. \tag{9}$$

With $\hat{\theta}(P)$ derived in equation (7), we can write the condition explicitly:

$$P_e = \frac{1}{2}\left(\frac{D_2 - D_1}{1 - qD_1/P_e} + \underline{\theta}\right). \tag{10}$$

Equation (10) has one and only one root in interval (D_1, D_2). We obtain the following closed-form solution for the equilibrium asset price P_e.[17]

$$P_e = \frac{(D_2 - D_1) + 2qD_1 + \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 + \underline{\theta}]^2 - 8qD_1\underline{\theta}}}{4} \tag{11}$$

[17] Details can be found in Appendix B.1.

For P_e to be an equilibrium, asset buyers should not have a profitable deviation by unilaterally bidding a higher price than P_e. That is, a buyer's expected payoff, $E[\theta|\theta < \hat{\theta}] - P$, should not increase in P. In the baseline model, the asset buyers' expected payoff takes the form

$$\pi(P) = \frac{1}{2}\left(\frac{D_2 - D_1}{1 - qD_1/P} + \underline{\theta}\right) - P.$$

For $P > D_1$, the expected payoff monotonically decreases in P.

$$\frac{d\pi(P)}{dP} = -\frac{qD_1(D_2 - D_1)}{2(1 - \frac{q}{P}D_1)^2 P^2} - 1 < 0$$

From (10), we know that the equilibrium asset price is such that $\pi(P_e) = 0$, so an asset buyer will earn negative profit if unilaterally bidding a higher price $P > P_e$. Intuitively, by bidding a higher price P, a buyer decreases her expected payoff in two ways. First, a higher bid increases the cost for acquiring a piece of asset, and directly reduces her payoff. Second, a higher price P also alleviates the bank run risk, making fewer banks sell for liquidity reasons. As a result, the buyer faces a pool of assets with deteriorating quality where more banks are selling assets because of fundamental insolvency. This again reduces her expected payoff.

Having solved P_e, we can obtain the corresponding equilibrium critical cash flow $\theta_e \equiv \hat{\theta}(P_e)$ from expression (9). One can also verify that $\theta_e \in (\theta^L, \theta^U)$.

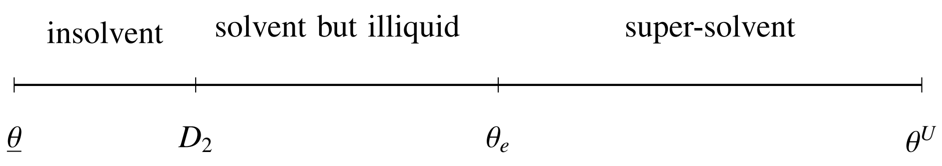
$$\theta_e = \frac{(D_2 - D_1) + 2qD_1 - \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 - \underline{\theta}]^2 + 4(D_2 - D_1)\underline{\theta}}}{2} \tag{12}$$

The market equilibrium $\{\theta_e, P_e\}$ reflects asymmetric information on asset qualities. By offering P_e, an uninformed buyer makes a loss when the bank is insolvent, and a profit when the bank is only illiquid. Furthermore, as a lower $\underline{\theta}$ aggravates the information asymmetry, it reduces the buyers' willingness to pay, and makes banks more likely to be illiquid. Mathematically, we have θ_e decreasing in $\underline{\theta}$.

Figure 3 illustrates the equilibrium funding liquidity risk. A bank with $\theta \in (D_2, \theta_e]$ may not fail and can fully repay its debt obligations if no bank run happens, yet it will fail because of premature asset liquidation caused by the run of its wholesale creditors.

Proposition 2. *The baseline model has an unique equilibrium, with equilibrium asset price P_e and equilibrium critical cash flow θ_e specified in* (11) *and* (12) *respectively. A bank with cash*

Figure 2: The unique equilibrium of the baseline model



flow $\theta \in (D_2, \theta_e)$ is solvent but illiquid: it will fail because of a wholesale debt run, even though its assets can generate a cash flow greater than its liabilities D_2.

Proof. See Appendix B.1. □

3.5 Application I: bank capital and bank run risk

It is conventional wisdom that capital helps reduces bank run risk. An application of the current framework, however, shows that the relationship is more subtle. We show that once asset prices are endogenous, capital also contributes to bank runs via stressed asset prices.

We model an increase of bank capital in its most simplistic form. We assume that a bank maintains its unit portfolio size, increasing its equity from E to $E + \Delta$, and at the same time decreasing its retail deposits from F to $F - \Delta$. In other words, an increase in capital reduces D_2 to $D_2 - \Delta$ but does not affect D_1. We then examine how increasing bank capital affects the risk of bank runs. To measure bank run risks, we follow Morris and Shin (2009) and define the illiquidity risk as $IL \equiv \hat{\theta}(P) - D_2$, with IL standing for illiquidity.[18]

Under exogenous asset prices, a natural corollary of Proposition 1 is that a higher capital always reduces funding liquidity risks, because the market value of capital serves as an extra buffer against fire-sale losses. The value of wholesale debt is better protected and wholesale creditors have less incentive to run, a channel that we call "buffer effect". Recall that $\hat{\theta}(P) = \frac{D_2 - D_1}{1 - \frac{q}{P} D_1}$, we can write IL explicitly as

$$IL = \frac{D_2 - D_1}{1 - \frac{q}{P} D_1} - D_2. \tag{13}$$

[18]Strictly speaking, the illiquidity risk should be measured as the probability $Prob(D_2 < \theta < \hat{\theta}(P)) = \frac{\hat{\theta}(P) - D_2}{\bar{\theta} - \underline{\theta}}$. We drop the the denominator because it is a constant and does not affect comparative statistics.

With price P exogenous and not a function of Δ, it is straightforward to verify that increasing bank capital unambiguously reduces illiquidity.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P - qD_1} < 0 \tag{14}$$

With endogenous asset prices, the situation is more complicated. Once investors rationally update their beliefs about a bank's asset qualities, a higher capital level also contributes to bank runs by reducing endogenous fire-sale prices. The intuition is as follows. In terms of inferring the realization of θ, a bank run presents more negative news when it happens to a well-capitalized bank than when it happens to a poorly capitalized bank. Because a well-capitalized bank is able to sustain large losses, the fundamentals of the bank must be unusually poor for a run to happen. With such pessimistic inference about θ, buyers' willingness to pay for the bank's assets decreases with the observed capital level. Therefore, a change in bank capital affects illiquidity not only via D_2 but also via the endogenous asset price P_e.

$$\frac{\partial IL}{\partial \Delta} = \frac{\partial IL}{\partial D_2}\frac{\partial D_2}{\partial \Delta} + \frac{\partial IL}{\partial P_e}\frac{\partial P_e}{\partial \Delta} \tag{15}$$

The first term captures the traditional "buffer effect" as in the case where the asset price is exogenous. The second term captures a new channel that we want to emphasize: increasing capital also affects banks' funding liquidity risk via endogenous asset prices.

To see that higher capital leads to lower secondary market asset prices, one can simply take the first order derivative of the closed-form solution of P_e, which gives

$$\frac{\partial P_e}{\partial \Delta} = -\frac{1}{4} - \frac{1}{4}\frac{D_1 + D_2 + \underline{\theta}}{\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}} < 0.$$

Increasing capital decreases asset buyers' willingness to pay for a bank's assets on sale, which in turn makes creditors panic and bank runs more likely. And this is captured by

$$\frac{\partial IL}{\partial P_e}\frac{\partial P_e}{\partial \Delta} > 0.$$

Hence, capital can contribute to funding liquidity risk by reducing endogenous asset prices, a mechanism we dub the "inference effect". Comparing expression (14) with (15), it should be clear that with endogenous asset prices and the "inference effect", capital is less able to contain bank run risks as compared to the case where asset price is exogenous. Buyers' rational beliefs limit the role of capital in containing funding liquidity risk.

The overall impact of capital on funding liquidity risk depends on the relative strength of the "buffer effect" and the "inference effect". Using the closed form solution of P_e and θ_e, one can write the overall impact of an increase in capital explicitly.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} + \frac{q(D_2 - D_1)D_1}{4(P_e - qD_1)^2}\left[\frac{1}{4} + \frac{D_1 + D_2 + \underline{\theta}}{4\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}\right] \tag{16}$$

It can be shown that in an extreme case where $\underline{\theta} = 0$, $\partial IL/\partial\Delta = 0$ and increasing capital cannot reduce funding liquidity risk at all. Intuitively, a lower $\underline{\theta}$ reduces the expected quality of assets on sale, and therefore reduces buyers' willingness to pay. That is,

$$\frac{\partial}{\partial \underline{\theta}}\left(\frac{\partial P_e}{\partial \Delta}\right) > 0.$$

The price drop is most pronounced when $\underline{\theta} = 0$. In that case, the "inference effect" reaches its maximum and completely offsets the "buffer effect" of capital. We summarize the results in the following proposition.

Proposition 3. *In equilibrium, higher bank capital leads to a lower fire-sale asset price. Compared to the case where the price is exogenous, capital is less able to reduce the risk of illiquidity. And in the extreme case where* $\underline{\theta} = 0$, *higher capital does not reduce bank illiquidity at all.*

Proof. See Appendix B.2 □

The result suggests that the design of prudential regulations has to take into account the responses of market participants. Compared to the situation where regulations are lax, market participants' interpretation of the same piece of negative news can be more pessimistic under stringent regulations. In this case, the effectiveness of stringent prudential regulations is reduced, or even completely eliminated.

4 Self-fulfilling bank runs and financial contagion

In this section, we extend the baseline model to include two banks and two states. Asset buyers will be able to update their beliefs about State s based on different numbers of bank runs.

They perceive $s = B$ to be more likely when more bank runs are observed. In the absence of commitment power, the equilibrium prices that buyers offer must reflect their posterior beliefs, and therefore vary with the number of runs. We characterize market equilibrium with a single bank run and that with two bank runs, respectively. We show that for a given $N \in \{1, 2\}$, there exists a unique market equilibrium characterized by $\{P_N^*, \theta_N^*\}$ (section 4.1 and 4.2). We further establish that financial contagion can arise as a multiple-equilibria phenomenon, highlighting how pessmistic beliefs can drive financial instability (section 4.3). Finally, we discuss how an asset purchase program committed by a regulator can improve financial stability over the market equilibria (section 4.4).

4.1 Market equilibrium with a single bank run

We start with characterizing the equilibrium with a single bank run. For a *given* asset price P_1 that corresponds to the single-bank-run outcome, the bank run game has a unique threshold equilibrium characterized by $\hat{\theta}(P_1)$. So asset buyers know that a bank run happens if and only if the bank's cash flow is lower than $\hat{\theta}(P_1)$, and update their beliefs about the aggregate state according to Bayes' rule. Recall that $\omega_1^s\left(\hat{\theta}(P_1)\right)$ denotes buyers' posterior belief for State s when they observe a single bank run.

$$
\begin{aligned}
\omega_1^B\left(\hat{\theta}(P_1)\right) \equiv Prob(s = B|N = 1) &= \frac{\left(\hat{\theta}(P_1) - \underline{\theta}_B\right)\left(\overline{\theta} - \hat{\theta}(P_1)\right)}{\left(\hat{\theta}(P_1) - \underline{\theta}_B\right)\left(\overline{\theta} - \hat{\theta}(P_1)\right) + \left(\hat{\theta}(P_1) - \underline{\theta}_G\right)\left(\overline{\theta} - \hat{\theta}(P_1)\right)} \\
&= \frac{\left(\hat{\theta}(P_1) - \underline{\theta}_B\right)}{\left(\hat{\theta}(P_1) - \underline{\theta}_B\right) + \left(\hat{\theta}(P_1) - \underline{\theta}_G\right)} \\
\omega_1^G\left(\hat{\theta}(P_1)\right) \equiv Prob(s = G|N = 1) &= \frac{\left(\hat{\theta}(P_1) - \underline{\theta}_G\right)\left(\overline{\theta} - \hat{\theta}(P_1)\right)}{\left(\hat{\theta}(P_1) - \underline{\theta}_B\right)\left(\overline{\theta} - \hat{\theta}(P_1)\right) + \left(\hat{\theta}(P_1) - \underline{\theta}_G\right)\left(\overline{\theta} - \hat{\theta}(P_1)\right)} \\
&= \frac{\left(\hat{\theta}(P_1) - \underline{\theta}_G\right)}{\left(\hat{\theta}(P_1) - \underline{\theta}_B\right) + \left(\hat{\theta}(P_1) - \underline{\theta}_G\right)}
\end{aligned}
$$

When the competitive asset market is in a rational expectations equilibrium, based on their posterior beliefs, asset buyers should expect to break even when purchasing bank assets for price P_1^*. Their ex-post zero-profit condition (8) can now be written as the following.

$$
P_1^* = E\left[\theta|\theta < \hat{\theta}(P_1^*), N = 1\right] = \omega_1^B\left(\hat{\theta}(P_1^*)\right)\frac{\underline{\theta}_B + \hat{\theta}(P_1^*)}{2} + \omega_1^G\left(\hat{\theta}(P_1^*)\right)\frac{\underline{\theta}_G + \hat{\theta}(P_1^*)}{2} \tag{17}
$$

A candidate equilibrium price P_1^* should be a fixed point of function $E\left[\theta|\theta < \hat{\theta}(P_1^*), N = 1\right]$. With $\theta_1^* \equiv \hat{\theta}(P_1^*)$, we can re-write the zero-profit condition (17) as a function of θ_1^*.[19]

$$F_1(\theta_1^*) \equiv \omega_1^B(\theta_1^*)\frac{\theta_B + \theta_1^*}{2} + \omega_1^G(\theta_1^*)\frac{\theta_G + \theta_1^*}{2} - \frac{qD_1\theta_1^*}{\theta_1^* - (D_2 - D_1)} = 0 \tag{18}$$

The equation simply states that asset buyers' *net* payoffs should be zero in expectation. And finding a fixed point P_1^* is equivalent to finding a solution for equation (18).

For P_1^* to be an equilibrium, an asset buyer must not profit by unilaterally raising her bid above P_1^*. In other words, function F_1 should not increase in P_1 (or equivalently, not decrease in θ_1). Such monotonicity always holds, and the intuition is as follows. First of all, as discussed in section 3.3, increasing price rises the cost of acquiring bank assets and leads to an deteriorating quality in the asset pool. Second, when P_1 increases, the risk of a bank run is mitigated, and a bank selling its assets is more likely to be fundamentally insolvent rather than facing a pure liquidity problem. For a *given* number of bank runs observed, this suggests that $s = B$ is more likely, i.e., $\partial\omega_1^B\left(\hat{\theta}(P_1)\right)/\partial P_1 > 0$. This further reduces the buyer's expected payoff.

Lemma 2. *$F_1(\theta)$ monotonically increases in θ_1, meaning that given a single bank run is observed, a buyer's expected payoff monotonically decreases in her bid P_1.*

Proof. See Appendix B.3. □

With extra complications introduced by the posterior beliefs on s, we can no longer obtain closed-form solutions for P_1^* and θ_1^*. Instead, we prove that there exists a $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$ that satisfies equation (18), and a corresponding $P_1^* \in (D_1, D_2)$ that satisfies equation (17). The proof is based on the continuity of $F_1(\theta_1)$. In particular, we show that $F_1(\theta_1)$ is negative at θ^L and positive at $\theta^U(P_1)$. Furthermore, given the monotonicity of $F_1(\theta)$, once an equilibrium exists, it is also unique. As a result, the market equilibrium with one bank run can be characterized by a unique pair $\{P_1^*, \theta_1^*\}$. The result is summarized in the proposition below.

Proposition 4. *There exist a unique equilibrium critical cash flow $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$ and a corresponding unique equilibrium asset price $P_1^* \in (D_1, D_2)$ associated with one bank run. A bank with cash flow $\theta \in (D_2, \theta_1^*]$ is solvent but illiquid.*

Proof. See Appendix B.4. □

[19] Here we have used the fact that $\theta_1^* \equiv \hat{\theta}(P_1^*) = \frac{(D_2 - D_1)}{1 - qD_1/P_1^*}$ so that $P_1^* = \frac{qD_1\theta_1^*}{\theta_1^* - (D_2 - D_1)}$.

4.2 Market equilibrium with two bank runs

Following the same approach as in the last section, we now characterize the equilibrium with two bank runs. For a *given* asset price P_2 that corresponds to a two-bank-run outcome, a bank will fail because of a run if and only if its cash flow $\theta < \hat{\theta}(P_2)$. Again, we formulate asset buyers' posterior beliefs about State s according to Bayes' rule.

$$\omega_2^B\left(\hat{\theta}(P_2)\right) \equiv Prob(s = B|N = 2) = \frac{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2}{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}$$
$$\omega_2^G\left(\hat{\theta}(P_2)\right) \equiv Prob(s = G|N = 2) = \frac{\left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}.$$

Based on the posterior beliefs, the asset buyers' break-even condition can be written as follows.

$$P_2^* = E\left[\theta|\theta < \hat{\theta}(P_2^*), N = 2\right] = \omega_2^B\left(\hat{\theta}(P_2^*)\right)\frac{\underline{\theta}_B + \hat{\theta}(P_2^*)}{2} + \omega_2^G\left(\hat{\theta}(P_2^*)\right)\frac{\underline{\theta}_G + \hat{\theta}(P_2^*)}{2} \tag{19}$$

And the equilibrium threshold $\theta_2^* \equiv \hat{\theta}(P_2^*)$ makes the following equation $F_2(\theta_2^*) = 0$.

$$F_2(\theta_2^*) \equiv \omega_2^B(\theta_2^*)\frac{\underline{\theta}_B + \theta_2^*}{2} + \omega_2^G(\theta_2^*)\frac{\underline{\theta}_G + \theta_2^*}{2} - \frac{qD_1\theta_2^*}{\theta_2^* - (D_2 - D_1)} = 0 \tag{20}$$

Lemma 3 shows that buyers' expected payoff monotonically decreases in P_2, so that they have no profitable deviation. Thus, any solution to equation (20) is indeed a market equilibrium.

Lemma 3. *$F_2(\theta)$ monotonically increases in θ, meaning that given two bank runs are observed, a buyer's expected payoff monotonically decreases in her bid P.*

Proof. See Appendix B.5. □

To prove the existence of and uniqueness of the equilibrium, we again use the monotonicity and continuity of function $F_2(\theta_2)$. We show that $F_2(\theta_2)$ is negative at θ^L and positive at θ^U, so that the market equilibrium with one bank run can be characterized by a unique pair $\{P_2^*, \theta_2^*\}$. The result is summarized in Proposition 5.

Proposition 5. *There exist a unique equilibrium critical cash flow $\theta_2^* \in (\theta^L, \theta^U(P_2^*))$ and a corresponding unique equilibrium asset price $P_2^* \in (D_1, D_2)$ associated with two bank runs. A bank with cash flow $\theta \in (D_2, \theta_2^*]$ is solvent but illiquid.*

Proof. See Appendix B.6. □

4.3 Financial contagion and multiple equilibria

$\theta_2^* > \theta_1^*$ would imply potential contagion. In particular, when a bank's cash flow lies between θ_1^* and θ_2^*, the bank will face no run if the other bank does not face a run, and will fail in a wholesale run if the other bank does. We prove with Lemma 4 that $\theta_2^* > \theta_1^*$ is indeed the case. Intuitively, the asset buyers form more pessimistic beliefs about State s when they observe more bank runs. Their willingness to pay for banks' assets decreases as banks' expected asset qualities are lower in State B. This in turn reduces equilibrium asset price and pushes up the equilibrium critical cash flow that a bank has to meet to survive a run.

Lemma 4. *When more runs are observed, the equilibrium market asset price is lower $P_2^* < P_1^*$ and the risk of bank runs is higher $\theta_2^* > \theta_1^*$.*

Proof. See Appendix B.7. □

Financial contagion emerges as a multiple-equilibrium phenomenon in the current model. In fact, when a bank's cash flow $\theta \in (\theta_1^*, \theta_2^*)$ and the other bank's cash flow $\theta < \theta_2^*$, the equilibrium number of bank runs depends on creditors' beliefs about each others' strategies. Only two threshold strategies can be be rationalized as part of a market equilibrium, i.e., an optimistic threshold strategy, 'to run if and only if $x < \theta_1^*$', and a pessimistic threshold strategy, 'to run if and only if $x < \theta_2^*$'. As a result, we can focus on those two threshold strategies only. We show that financial contagion can happen purely because of creditors' pessimistic beliefs.

For the ease of exposition, we label the two banks as Bank i and j, and discuss the following two cases respectively. (1) Bank i has a cash flow $\theta \in (\theta_1^*, \theta_2^*)$ and Bank j has a cash flow $\theta < \theta_1^*$.[20] And (2) Bank i and j both have cash flows between θ_1^* and θ_2^*.

In the first case, the equilibrium number of bank runs can be either 1 or 2, depending on creditors' belief about each others' strategies. With a cash flow $\theta < \theta_1^*$, Bank j will fail in a run whether creditors follow the optimistic or pessimistic strategy. Therefore, there will be at least one bank run in the economy. Whether Bank i will have a run, however, depends on creditors' beliefs. If creditors believe that a positive mass among them follow the pessimistic strategy, they will expect a run on Bank i and an asset price P_2^*, so that it is optimal to join the run. As a result, all creditors withdrawing early from Bank i can emerge as an equilibrium. By contrast, if all creditors believe that none of them follow the pessimistic strategy, they would expect the

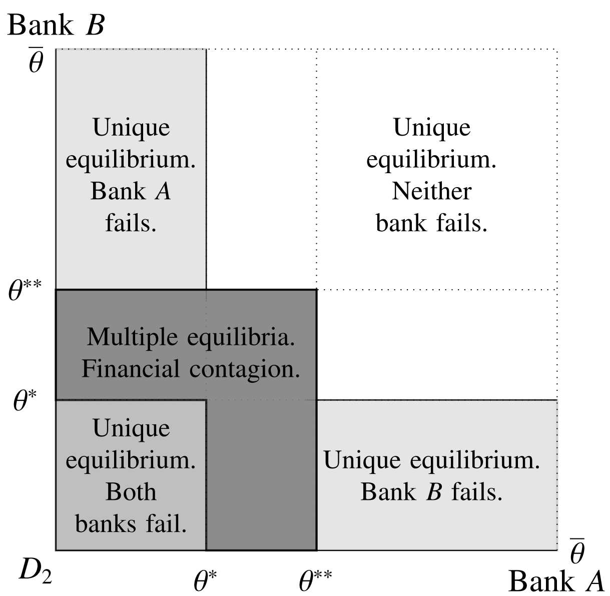
[20] The symmetric case where Bank i has a cash flow $\theta < \theta_1^*$, and Bank j has $\theta \in (\theta_1^*, \theta_2^*)$ can be analyzed with the same reasoning.

asset price to be P_1^*, and only Bank j to fail, which justifies their optimistic belief/strategy in the first place.

In the second case, the equilibrium number of bank runs can be either 0 or 2, depending again on creditors' beliefs. If all creditors believe that none of them follow the pessimistic strategy, no run will happen, because both banks' cash flows are higher than θ_1^*. Therefore, $N = 0$ can be an equilibrium. On contrast, if a creditor believes that a positive mass among them follow the pessimistic strategy, he will expect two bank runs and assets sold for price P_2^*, so that it is optimal for him to join the run. Therefore, $N = 0$ can emerge as an equilibrium. The creditor's belief must be that a positive mass of creditors will run both banks. This is because if the pessimistic creditors are present in one bank, then those creditors' strategy cannot be rationalized. Therfore, $N = 1$ cannot be an equilibrium.

In sum, multiple equilibria can emerge when one bank's cash flow is in $[\theta_1^*, \theta_2^*]$ and the other bank's cash flow is below θ_2^*. The contagion is self-fulfilling and can be fuelled completely by creditors' beliefs. In Figure 4, we plot the possible equilibrium outcomes for different combinations of bank cash flows, and summarize the results in Proposition 6.

Figure 3: Equilibrium of the fully-fledged model



Proposition 6. *When one bank's cash flow belongs to* $[\theta_1^*, \theta_2^*]$ *and the other bank's cash flow is lower than* θ_2^**, multiple market equilibria exist, and financial contagion can happen because of creditors' pessimistic beliefs.*

4.4 Application II: asset purchase program

We show in this section that a regulator with commitment power can promote financial stability even if he is not better informed than the asset buyers. The welfare-improving policy intervention that we propose resembles asset purchase programs such as Term Asset-Backed Securities Loan Facility (TALF).

We consider the following policy intervention: the regulator makes a promise to purchase bank assets at a *unified* price P_A in case any bank run happens. In particular, the unified price P_A does not depend on the number of bank runs in the economy. The regulator is assumed to have full commitment power and will not revoke his offer after having observed the actual number of bank runs. Under the policy intervention, the model has the following revised timeline.

Figure 4: Timing of the asset purchase program

t = 0	t = 0.5	t = 1	t = 2
Banks are established, with their portfolios and liability structures as given.	The regulator makes a promises to buy assets at a unified price P_A, in case any bank run happens.	1. s and θ are realized. 2. Asset buyers post a price scheme. 3. For each bank that they lend to, creditors receive private noisy signals about the bank's cash flow θ, and decide to run or not. 4. Assets are sold to the party that offers the highest price.	1. Returns are realized. 2. Remaining obligations are settled

The regulator is risk-neutral, and is subject to an ex-ante budget constraint: he should not make any loss in expectation. To maximise social welfare, he will choose an optimal price P_A^* so that he only breaks even. This is because any lower price that leads to a positive expected profit will come at a cost of letting more solvent banks fail in runs.

The regulator is different from the typical asset buyers in the market because he holds full commitment power. In particular, he is not required to break even for each observed number of bank runs, but only to break even ex ante. The commitment power allows the regulator to disregard new information such as the number of bank runs, and can therefore avoid the vicious cycle fuelled by pessimistic belief updating.

We now derive the ex-ante break-even price P_A^*. As a first step, we solve for price P_A^*, under the assumption that banks will sell their assets to the regulator instead of to the asset buyers in the secondary market. When wholesale creditors expect bank assets to be sold at price P_A, we know from Section 3.3 that the critical cash flow of the bank run game is $\hat{\theta}(P_A)$. So the regulator understands that only those assets with $\theta < \hat{\theta}(P_A)$ will be on sale, with $\hat{\theta}(P_A)$ again defined by expression (7).

$$\hat{\theta}(P_A) = \frac{D_2 - D_1}{1 - qD_1/P_A} \tag{21}$$

As the regulator commits to price P_A before observing any number of bank runs, he holds the prior belief $Prob(s = G) = Prob(s = B) = 1/2$. From this ex-ante perspective, the regulator's break-even condition can be written as follows.

$$P_A^* = \frac{1}{2}\frac{\underline{\theta}_B + \hat{\theta}(P_A^*)}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \hat{\theta}(P_A^*)}{2} \tag{22}$$

Using expression (21), we can rewrite the ex-ante break-even condition (8) into a quadratic function of P_A^*, which has the following root between D_1 and D_2.

$$P_A^* = \frac{[2(D_2-D_1)+4qD_1+(\underline{\theta}_B+\underline{\theta}_G)] + \sqrt{[2(D_2-D_1)+4qD_1+(\underline{\theta}_B+\underline{\theta}_G)]^2 - 16qD_1(\underline{\theta}_B+\underline{\theta}_G)}}{8} \tag{23}$$

Having obtained P_A^*, we can derive the corresponding $\hat{\theta}(P_A^*)$ using equation (21). Following the same procedure as in the proof of Proposition 4, we can prove that $\hat{\theta}(P_A^*) \in (\theta^L, \theta^U(P_A^*))$, so that the policy intervention cannot completely eliminate inefficient bank runs.

Lemma 5. *Suppose that facing runs, banks can sell their assets at a unified price committed by the regulator. The regulator can break even ex ante by offering price P_A^* as in (23). And the bank run game has a threshold equilibrium where a run happens if and only if $\theta < \hat{\theta}(P_A^*)$.*

Proof. See Appendix B.8. □

Now one can verify that P_A^* is higher than what market offers (i.e., $P_A^* > P_1^* > P_2^*$), so that banks will indeed sell their assets to the regulator. As $\hat{\theta}(P)$ decreases with P, the policy intervention improves financial stability as compared to the market equilibria. In particular, the asset purchase committed by the regulator reduces (though does not eliminate) the risk of bank runs, and completely rules out financial contagion. The result is summarised in Proposition 7.

Proposition 7. *The regulator's ex-ante break-even price P_A^* is higher than the prices in market equilibria, so that banks will sell their assets to the regulator when they face runs. With P_A^* >*

$P_1^* > P_2^*$ and $\hat{\theta}(P_A^*) < \hat{\theta}(P_1^*) < \hat{\theta}(P_2^*)$, the regulator can reduce bank run risks and eliminate financial contagion.

Proof. See Appendix B.9. □

With his commitment power, the regulator can disregard the outcome of bank run games and stick to a unified asset price. The commitment power allows the regulator to avoid the vicious cycle between bank runs and fire sales that is fuelled by pessimistic beliefs in market. As the regulator only needs to break even ex ante given his prior belief about State s, he can use the profit from State G to compensate the loss in State B.

The typical buyers in market, on the other hand, are unable to do so. Without commitment power, they must not make expected loss given any realized number of bank runs. In other words, they are constrained by ex-post break-even conditions. In fact, if an asset buyer offers the same price P_A^*, she will revoke the offer when a bank run actually happens, because in that case she will form a posterior belief that $s = B$ is more likely and will no longer consider herself breaking even by purchasing bank assets at P_A^*. To break even from this ex-post perspective, the asset buyer has to lower her offered price, so as to decrease the loss from purchasing assets with $\theta \in [\underline{\theta}_s, P_N^*)$, and to increase the profit from purchasing assets with $\theta \in \left[P_N^*, \hat{\theta}(P_N^*)\right)$. The lack of commitment power therefore leads to lower asset prices, which in turn result in more bank runs, and justify the pessimistic beliefs in the first place.

This result naturally relates to Lender of Last Resort (LOLR) policies. The effectiveness of such policies has long been debated, because a lender of last resort may not be able to tell whether a troubled bank is insolvent or illiquid. As a result, blinded intervention will compromise market discipline, whereas taking no action runs the risk of letting solvent banks fail. The current model, however, shows that interventions such as asset purchase programs can still improve financial stability without demanding information on individual banks' financial healthiness. Even if such interventions is not perfect—banks with $\theta \in (D_2, \hat{\theta}(P_A))$ still fail because of illiquidity, the dilemma is not as stark as one may think.

5 Further policy discussion

The model is sufficiently rich for other policy analysis, and we present here one more policy implication for regulatory disclosure.[21] We focus on a situation where a regulator has superior information about aggregate state s and can credibly disclose the information to the market. We analyze the cost and benefit of such regulatory disclosure, and compare it with asset purchase programs in terms of promoting financial stability.

5.1 Trade-offs for regulatory disclosure

To concentrate on the effects of disclosure, we consider a simplistic case where the regulator observes State s perfectly. Once the regulator decides to disclose the information, the realization of State s will be released before market trading. We also assume that the regulator can commit to truthful revelations by legislation. The information set of asset buyers changes correspondingly. Instead of updating beliefs about State s based on the number of bank runs, buyers now learn the state with certainty. Therefore, asset prices can be conditional on the true state that the regulator discloses.

The regulatory disclosure eliminates alternative beliefs as a source of multiple equilibria. Instead of two rational expectations equilibria depending on buyers' beliefs, there will be a unique equilibrium for each disclosed (realized) state. For brevity, we spare the derivation of market equilibria, and denote by $\{\theta^*_G, P^*_G\}$ and $\{\theta^*_B, P^*_B\}$ the equilibrium critical cash flows and asset prices for State G and B respectively. Lemma 6 demonstrates the effect of regulatory disclosure on banks' illiquidity.

Lemma 6. *When the aggregate State s is disclosed to market participants, there exists a unique market equilibrium $\{\theta^*_s, P^*_s\}$ associated with each realized State $s \in \{G, B\}$. In State s, a bank with cash flow $\theta \in (D_2, \theta^*_s]$ will fail because of illiquidity. Such regulatory disclosure eliminates the multiple equilibria caused by the asset buyers' beliefs about the aggregate state.*

Proof. See Appendix B.10. □

Intuitively, we have $\theta^*_B > \theta^*_2$ and $\theta^*_G < \theta^*_1$. Here $\theta^*_B > \theta^*_2$ because even when observing two bank runs, buyers cannot exclude the possibility of $s = G$. But to the extent that a regulatory disclosure $s = B$ is accurate, buyers will offer price according to $s = B$ with certainty. Similarly,

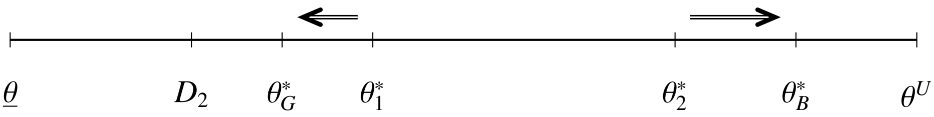
[21] In reality, examples of the regulatory disclosure include communicating stress testing parameters to the public or making announcements of the size of assistance programs.

making a favorable disclosure $s = G$ further reassures market participants and can save banks from illiquidity. The result is summarized in Proposition 8 and is illustrated by Figure 5.

Proposition 8. *With $\theta_G^* < \theta_1^*$ and $\theta_B^* > \theta_2^*$, the regulatory disclosure reduces illiquidity if $s = G$ but increases it if $s = B$.*

Proof. See Appendix B.11. □

Figure 5: Equilibrium under regulatory disclosure



The disclosed information, when favorable, boosts asset prices and saves banks with $\theta \in (\theta_G^*, \theta_1^*]$ from illiquidity. Acknowledging a bad state, on the other hand, exacerbates liquidity problems. That is, a solvent bank is more likely to suffer from illiquidity when market participants are aware of the downside risk, and banks with $\theta \in (\theta_2^*, \theta_B^*]$ will fail because of runs. Therefore, in determining whether to disclose information to the public, regulators face a trade-off: if the state is good, the reassuring disclosure can save banks from illiquidity; but if the state is unfavorable, acknowledging a crisis will create even more runs by pushing asset prices further down. Arguably, when the social cost of bank failure is greater in State B, it is suboptimal for regulators to commit to disclosing information.

5.2 Regulatory disclosure vs. Asset purchase

Now we run a horse race between different policy interventions, examining whether regulatory disclosure can outperform an asset purchase program as modelled in section 4.4. The regulator is assumed to choose between the two policy interventions before State s realizes. We show that an asset purchase program, which does not require superior information on the aggregate state, can actually achieve a higher level of financial stability.

Note first that asset purchase programs and regulatory disclosure are mutually exclusive. Once it has been credibly communicated that $s = G$, the equilibrium asset price will be P_G^*—higher than the break-even price P_A^* in asset purchase programs. The reason is again that asset buyers are most optimistic when they learn the state is good with certainty. Suppose that an asset purchase program and regulatory disclosure coexist, the regulator will not acquire any assets in the good state unless his announced price is higher than P_G^*. Such a price, however, implies making losses from an ex-ante perspective.

We then evaluate the regulator's policy choice between asset purchase programs and regulatory disclosure. For simplicity, we concentrate on the social cost of bank failures. We denote the social cost by C for each failed bank and assume it is independent of the number of bank failures and State s. The regulator aims to choose a policy intervention that minimizes the expected social cost. We denote by SC^{AP} and SC^{RD} the expected social costs associated with asset purchase programs and the regulatory disclosure, respectively. Recall from section 4.4 that $\hat{\theta}(P_A^*)$ is the critical cash flow for the regulator's ex-ante break-even price P_A^*. SC^{AP} can be formulated as

$$SC^{AP} = \frac{1}{2} \cdot \frac{\hat{\theta}(P_A^*) - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} C + \frac{1}{2} \cdot \frac{\hat{\theta}(P_A^*) - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} C. \tag{24}$$

Whereas the expected social cost associated with regulatory disclosure can be written as

$$SC^{RD} = \frac{1}{2} \cdot \frac{\hat{\theta}(P_B^*) - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} C + \frac{1}{2} \cdot \frac{\hat{\theta}(P_G^*) - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} C. \tag{25}$$

We show with Corollary 1 that the social cost associated with asset purchase programs is strictly lower. This result stems from the fact that the critical cash flow $\hat{\theta}(P)$ is decreasing and convex in P. The result suggests that the economy would be better off if the regulator disregards his superior information about State s, and commits to purchase bank assets at price P_A^*.

Corollary 1. *The expected social cost due to bank failures is lower under asset purchase programs than under regulatory disclosure,* $SC^{AP} < SC^{RT}$.

Proof. See Appendix B.12. □

6 Concluding remarks

In this paper, we have investigated the relationship between fire-sales and bank runs. We have presented a model where asset fire sales and bank runs are driven by the lack of information and endogenously determined in a rational expectations equilibrium. We have also extended the model to incorporate contagion when there is a common risk exposure. We can draw several conclusions from our analysis. First, asset fire sales and bank runs are self-fulfilling and mutually reinforcing: when creditors anticipate low prices for a bank's assets, a run will be triggered, which generates fire-sales and the corresponding collapse in prices, thus fully justifying creditors' strategies. Second, as one bank fails, asset buyers lower their expectations of the common risk factor and perceive banks' assets to be less valuable: the declining asset prices will precipitate contagious runs at all other banks.

The model has derived three policy implications regarding bank capital, asset purchase programs and regulatory disclosure. First, contrary to the conventional wisdom, we have shown that bank capital holding can have unintended consequences for funding liquidity, because a run on a well-capitalized bank signals unusually high risk and exacerbates asset fire sales, which in turn makes the run more likely. In its extremity, the model predicts that capital cannot reduce bank funding liquidity risk at all. Second, we have demonstrated that a regulator can break down the vicious circle between asset fire sales and bank runs by committing to purchase banks' assets at a predetermined price. This contributes to the discussion of asset purchase programs: by implementing such programs, regulators can promote financial stability and still break even from ex-ante perspective even with no better information than other market participants. Finally, we have shown that regulatory disclosure is a double-edged sword. It saves banks from illiquidity when the disclosure is favorable. But, it amplifies funding liquidity risk and financial contagion when the disclosure worsens market beliefs.

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Appendix A Bank run game for $D_1 < P_N < D_2$

In this section, we solve the creditors' bank run game for a given secondary market price P_N belongs to the interval (D_1, D_2).

Appendix A.1 Lower and upper dominance regions

Following the standard procedure of global games, we start with the lower dominance region denoting as $[\underline{\theta}, \theta^L]$. Suppose all other creditors stay until $t = 2$ when the bank's cash flow realizes in this region, then the fraction of creditors who withdraw is $L = 0$. Under this case, there is no bank run. In this circumstance, a creditor i still withdraws at $t = 1$ if and only if the inequality (5) holds for $L = 0$, that is $\theta \leq F + (1 - E - F)r_D = D_2$. The bank's fundamental is so poor that the bank still fails at $t = 2$ even if there is no premature liquidation of its assets. A creditor i who waits will get zero because of the bankruptcy. Instead, he will get qr_D if withdrawing early. Thus, we define $\theta^L = D_2$. In our analysis, the support of noise ϵ is taken to be arbitrarily close to zero, so creditors are sure when the bank's cash flow realizes in $[\underline{\theta}, D_2)$. Thus, a creditor's dominant strategy is to withdraw at $t = 1$ to get qr_D in this circumstance.

Second, we choose $\theta^U(P_N) = \frac{F}{1-D_1/P_N}$ given the asset price P_N. Then the upper dominance region is $[\theta^U(P_N), \overline{\theta}]$. Note that we can always have

$$\overline{\theta} > \frac{F}{1 - D_1/P_N}$$

by assuming $\overline{\theta}$ is sufficiently large to keep the existence of the upper dominance region. Now suppose all other creditors withdraw early when the bank's fundamental realizes in the upper dominance region, then $L = 1$. Under this case, a successful bank run always occurs irrespective of the bank's cash flow. Yet, the bank still survives at $t = 2$ if the inequality (5) does not hold, $(1 - D_1/P_N)\theta > F$. In other words, the bank always survives if its realized cash flow is sufficiently large $\theta > \frac{F}{1-D_1/P_N}$. Again, the creditors' signals are arbitrarily accurate, they are sure when the bank's cash flow realizes in $(\theta^U(P_N), \overline{\theta}]$. A creditor's dominant strategy is to stay until $t = 2$ to avoid the penalty for early withdrawal ($r_D > qr_D$).

Appendix A.2 Beliefs of creditors outside the dominance regions

In this subsection, we characterize creditors' beliefs when the bank's cash flow realizes in the intermediate region $(\theta^L, \theta^U(P_N))$. Now creditors' actions depend on their beliefs about the actions of other creditors. The signals regarding to the bank's realized cash flow form their beliefs.

To proceed, we first determine the fraction of creditors who withdraw at $t = 1$ as a function of a bank's realized cash flow and the threshold. Formally, when a bank's cash flow θ realizes in the region $(\theta^L, \theta^U(P_N))$, a creditor i receives a signal $x_i = \theta + \epsilon_i$, with $\epsilon_i \sim U(-\epsilon, \epsilon)$ as the noise about the realized fundamental. We suppose each creditor acts according to a threshold strategy and set the threshold signal as $\hat{x}$, i.e., a creditor i withdraws at $t = 1$ if $x_i < \hat{x}$, stays until $t = 2$ if $x_i > \hat{s}$. The fraction of creditors who withdraw at $t = 1$ should be a function of the realized cash flow θ and the threshold of signals $\hat{x}$, that is $L = L(\theta, \hat{x})$. This is because the decision to withdraw or stay depends on both the realization of the cash flow and the strategy of other players. To achieve model tractability, we follow the classic approach in global games by assuming that the creditors' signal about the realized cash flow is sufficiently accurate. The noise ϵ_i is distributed on an arbitrarily small interval, $\epsilon \to 0$. As a result, we can consider the threshold of signal $\hat{x}$ approximately to be a threshold of bank's cash flow $\hat{\theta}$, as $\hat{x}$ and $\hat{\theta}$ are arbitrarily close. Then a representative creditor i withdraws at $t = 1$ if $x_i < \hat{\theta}$, stays till $t = 2$ if $x_i > \hat{\theta}$ and the fraction of early withdrawals is $L(\theta, \hat{\theta})$.

Our second step is to determine the functional form of $L(\theta, \hat{\theta})$. For a realized θ, we have three cases: (i) When $\theta + \epsilon < \hat{\theta}$, even the highest possible signal is below the threshold $\hat{\theta}$. According to the definition of threshold strategy, all creditors withdraw at $t = 1$, and $L(\theta, \hat{\theta}) = 1$. (ii) When $\theta - \epsilon > \hat{\theta}$, even the lowest possible signal exceeds the threshold $\hat{\theta}$. Then all creditors stay till $t = 2$. (iii) When θ falls into the intermediate range $[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]$, the fraction of creditors who withdraw at $t = 1$ is determined as

$$L(\theta, \hat{\theta}) = Prob(x_i < \hat{\theta}|\theta) = Prob(\epsilon_i < \hat{\theta} - \theta|\theta) = \frac{\hat{\theta} - \theta - (-\epsilon)}{2\epsilon} = \frac{\hat{\theta} - \theta + \epsilon}{2\epsilon}. \tag{A.26}$$

A creditor who receives a signal x_i holds a posterior belief that the fundamental follows a uniform distribution on $[x_i - \epsilon, x_i + \epsilon]$ because the noise ϵ_i is uniformly distributed on $[-\epsilon, \epsilon]$. As the proportion of creditors who withdraw is a function of the fundamental, each creditor forms a posterior belief about the proportion.

The third step is to derive those posterior beliefs. To begin with, we show that the distribution is uniform on $[0, 1]$ for the marginal creditor who happens to observe $s_i = \hat{\theta}$. Indeed, we have

$$Prob\left(L(\theta,\hat{\theta}) \leq \hat{L}\middle|x_i = \hat{\theta}\right) = Prob\left(\frac{\hat{\theta}-\theta+\epsilon}{2\epsilon} \leq \hat{L}\middle|x_i = \hat{\theta}\right) = Prob\left(\theta \geq \hat{\theta}+\epsilon-2\epsilon\hat{L}\middle|x_i = \hat{\theta}\right).$$

On the other hand, we know that, conditional on $x_i = \hat{\theta}$, the marginal creditor has a posterior belief that θ is uniformly distributed on $[\hat{\theta}-\epsilon, \hat{\theta}+\epsilon]$, which implies $Prob\left(L(\theta,\hat{\theta}) \leq \hat{L}\middle|x_i\right) = \hat{L}$. Therefore, the marginal creditor holds a posterior belief that the fraction of creditors who withdraw at $t = 1$ forms a uniform distribution on $[0, 1]$, that is $L(\theta, \hat{\theta}|x_i = \hat{\theta}) \sim U(0, 1)$.

We then move onto the slightly more complicated cases for the non marginal creditor, $x_i \neq \hat{\theta}$. Without loss of generality, we start with the case $x_i > \hat{\theta}$. Remember that a creditor who receives a signal x_i holds a posterior belief that the fundamental follows a uniform distribution on $[x_i - \epsilon, x_i + \epsilon]$. Given $x_i > \hat{\theta}$, the upper bound of the support is greater than $\hat{\theta}+\epsilon$. And we know that when $\theta > \hat{\theta}+\epsilon$, all creditors stay and $L = 0$. In fact, we can divide the support of θ into two sections: $[x_i - \epsilon, \hat{\theta}+\epsilon]$ and $[\hat{\theta}+\epsilon, x_i+\epsilon]$. As we have discussed, the second section corresponds to a posterior belief $L(\theta, \hat{\theta}|x_i) = 0$. Therefore in the eyes of a creditor i who receives $x_i > \hat{\theta}$, there will be a positive probability mass on $L = 0$. On the other hand, we can show that the posterior belief of θ continues to be a uniform distribution on $[x_i - \epsilon, \hat{\theta}+\epsilon] \subset [\hat{\theta}-\epsilon, \hat{\theta}+\epsilon]$. Since θ is again within the intermediate range $[\hat{\theta}-\epsilon, \hat{\theta}+\epsilon]$, the expression of $L(\theta, \hat{\theta})$ will follow expression (A.26), and we can derive the posterior belief on L as follows.

$$Prob\left(L(\theta,\hat{\theta}) \leq \hat{L}|x_i\right) = Prob\left(\frac{\hat{\theta}-\theta+\epsilon}{2\epsilon} \leq \hat{L}\middle|x_i\right) = Prob(\theta \geq \hat{\theta}+\epsilon-2\epsilon\hat{L}|x_i)$$

Because the player perceives a uniform distribution of θ on $[x_i - \epsilon, \hat{\theta}+\epsilon]$, the probability above can be calculated as $\frac{\hat{L}}{1-(s_i-\hat{\theta})/2\epsilon}$, and this is a uniform distribution on $\left[0, 1-\frac{x_i-\hat{\theta}}{2\epsilon}\right]$. Notice that the density function on this interval is 1, thus the probability uniformly allocated on this interval is $1-\frac{x_i-\hat{\theta}}{2\epsilon}$, and the probability mass at $L(\theta,\hat{\theta}) = 0$ is $Prob\left(L(\theta,\hat{\theta}) = 0\right) = \frac{x_i-\hat{\theta}}{2\epsilon}$. A creditor who observes $x_i > \hat{\theta}$ holds a more optimistic belief that a smaller proportion of creditors will withdraw (reflected by the positive probability mass on $L = 0$ where no one withdraws). As the marginal creditor who observes $x_i = \hat{\theta}$ is indifferent between withdrawing or not, the player who observes $x_i > \hat{\theta}$ will prefer to stay. Moreover, the higher the signal s_i received, the more optimistic belief a creditor i holds ($Prob\left(L(\theta,\hat{\theta}) = 0\right) = \frac{x_i-\hat{\theta}}{2\epsilon}$ increases in x_i).

The case $x_i < \hat{\theta}$ follows exactly the same procedure. We can show that from the perspective of a creditor who observes $x_i < \hat{\theta}$, L has a mixed distribution: it is uniformly distributed on $\left[\frac{\hat{\theta}-x_i}{2\epsilon}, 1\right]$ with density function 1, and has with a positive probability mass at $L(\theta, \hat{\theta}) = 1$. The probability mass at $L = 1$ is $Prob\left(L(\theta, \hat{\theta}) = 1\right) = \frac{\hat{\theta}-x_i}{2\epsilon}$, where creditor i believes every one withdraws. Thus, a creditor who observes $x_i < \hat{\theta}$ will be more pessimistic and prefer to withdraw. Moreover, the lower the signal s_i received, more pessimistic belief a creditor i holds ($Prob\left(L(\theta, \hat{\theta}) = 1\right) = \frac{\hat{\theta}-x_i}{2\epsilon}$ increases in x_i).

Appendix A.3 Threshold Equilibrium

The previous subsections show that upper and lower dominance regions are existent and any creditor whose signal is higher (lower) than $\hat{\theta}(P_N)$ is more prone to stay (withdraw). Now we formally derive the value of this critical cash flow by the indifference condition of the marginal creditor. Remember that the marginal creditor, observing exactly $\hat{\theta}$, is indifferent between stay and withdraw. We have derived that his belief is $L \sim U(0, 1)$ and formulated the difference $DW(L)$ in Section 3.2. Then the creditor's indifference condition can be expressed as

$$\int_0^1 DW(L)dL = 0,$$

or

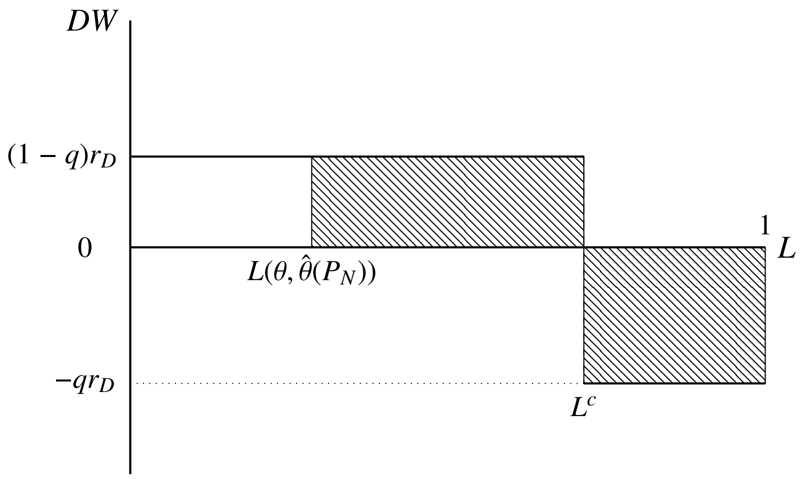
$$\int_{L^c}^1 qr_D dL - \int_0^{L^c} (1-q)r_D dL = qr_D(1-L^c) - (1-q)r_D L^c = 0.$$

Recall the definition of L^c, $L^c = \frac{P_N(\theta-D_2)}{(\theta-P_N/q)D_1}$. The indifference condition implies a unique critical cash flow $\hat{\theta}$ for a given asset price $P_N \in (D_1, D_2)$.

$$\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N},$$

For a given asset price $P_N \in (D_1, D_2)$, a run happens to banks with $\theta < \hat{\theta}(P_N)$. Geometrically, we present the indifference condition in Figure 3.

Figure 6: Payoff differences and the decision to withdraw



Appendix B Proofs to Lemmas and Propositions

Appendix B.1 Proposition 2. Solution to the baseline model

Proof. To solve the equilibrium critical cash flow θ_e, note that (**??**) is actually a quadratic function of $\hat{\theta}$

$$\hat{\theta}^2 - \left[(D_2 - D_1) + 2qD_1 - \underline{\theta}\right]\hat{\theta} - (D_2 - D_1)\underline{\theta} = 0.$$

Using the quadratic formula, we can obtain two solutions and retain the positive one

$$\theta_e = \frac{(D_2 - D_1) + 2qD_1 - \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 - \underline{\theta}]^2 + 4(D_2 - D_1)\underline{\theta}}}{2}.$$

The equilibrium asset price P_e can be obtained by solving (10) or directly from the zero profit condition $P_e = \frac{\theta_e + \underline{\theta}}{2}$. We have

$$P_e = \frac{(D_2 - D_1) + 2qD_1 + \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 + \underline{\theta}]^2 - 8qD_1\underline{\theta}}}{4},$$

Note that $[(D_2 - D_1) + 2qD_1 - \underline{\theta}]^2 + 4(D_2 - D_1)\underline{\theta} = [(D_2 - D_1) + 2qD_1 + \underline{\theta}]^2 - 8qD_1\underline{\theta}$.

To prove $\theta_e > D_2$ and $P_e \in (D_1, D_2)$, we let q be sufficiently close to 1 to simplify the calculation. Note that this assumption is innocuous as q is the penalty for early withdrawal, in reality such penalty is small for demandable debts, i.e., $q \to 1$. So θ_e and P_e turn into

$$\theta_e = \frac{(D_1 + D_2 - \underline{\theta}) + \sqrt{[(D_1 + D_2 - \underline{\theta})^2 + 4F\underline{\theta}}}{2}, \quad P_e = \frac{(D_1 + D_2 + \underline{\theta}) + \sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8qD_1\underline{\theta}}}{4}.$$

With the analytical solution, it can be verified easily that $D_1 < P_e < D_2$ and $\theta_e > D_2$. To prove $\theta_e < \theta^U(P_e)$, note that $\theta_e = \frac{D_2 - D_1}{1 - qD_1/P_e}$ and $\theta^U(P_e) = \frac{F}{1 - D_1/P_e}$. With $P_e > D_1$, $\theta^U(P_e)$ is finite, thus can be assumed to be less than $\overline{\theta}$. By the definition of D_2 and D_1, we have $\theta_e = \theta^U(P_e)$ when $q \to 1$. Then consider the following derivative

$$\begin{aligned} lim_{q \to 1^-} \frac{d}{dq} \left[\theta^U(P_e) - \theta_e\right] =& lim_{q \to 1^-} \frac{d}{dq} \left[\frac{F}{1 - \frac{q(1-E-F)r_D}{P_e}} - \frac{F + (1-q)(1-E-F)r_D}{1 - \frac{q^2(1-E-F)r_D}{P_e}} \right] \\ =& \frac{P_e - D_2}{P_e}. \end{aligned}$$

Thus, there exists an interval for q such that when $q \in (1-\epsilon, 1)$, $\frac{d}{dq}\left(\theta^U(P_e) - \theta_e\right) < 0$ if and only if $P_e < D_2$. Combining with $\theta_e = \theta^U(P_e)$ when $q = 1$, we obtain $\theta^U(P_e) > \theta_e$ when $q \in (1-\epsilon, 1)$. That is $\theta_e \in [D_2, \theta^U(P_e)] \subset [D_2, \overline{\theta}]$.

To conclude, θ_e and P_e derived above is the unique equilibrium critical cash flow and asset price in the baseline model. Thus, the creditors' beliefs and asset buyers' beliefs are consistent.

□

Appendix B.2 Proposition 3. Bank capital and illiquidity

Proof. We show that increasing capital is less able, or even has no effect in reducing a bank's illiquidity risk when asset price is endogenous.

From (16), we obtain

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} \frac{(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}} - (D_2 - D_1)P_e}{(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}$$

Provided that $P_e > D_1$, we have $sgn\left(\frac{\partial IL}{\partial \Delta}\right) = -sgn\left[(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}} - (D_2 - D_1)P_e\right]$. With the analytical form of P_e from Appendix B.2, we have

$$sgn\left[(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}} - (D_2 - D_1)P_e\right] = sgn\left[D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right]$$

When $\underline{\theta} = 0$, it can be further verified that $P_e = \frac{D_1+D_2}{2}$ and $D_1(D_2-P_e)+(\underline{\theta}-D_1)(P_e-D_1) = 0$. Thus, we obtain $sgn\left(\frac{\partial IL}{\partial \Delta}\right) = 0$. In this case, increasing capital (increase Δ) has no effect on a bank's illiquidity risk.

When $\underline{\theta} > 0$, we take the derivative $\frac{\partial}{\partial \underline{\theta}}\left[D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right] = (\underline{\theta} - 2D_1)\frac{\partial P_e}{\partial \underline{\theta}} + (P_e - D_1)$. Recall again $P_e = \frac{(D_1+D_2+\underline{\theta})+\sqrt{(D_1+D_2+\underline{\theta})^2-8D_1\underline{\theta}}}{4}$, we can calculate $\frac{\partial P_e}{\partial \underline{\theta}} = \frac{P_e-D_1}{\sqrt{(D_1+D_2+\underline{\theta})^2-8D_1\underline{\theta}}}$. Thus, we have

$$(\underline{\theta} - 2D_1)\frac{\partial P_e}{\partial \underline{\theta}} + (P_e - D_1) = (P_e - D_1)\frac{\underline{\theta} - 2D_1 + \sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}{\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}.$$

As $P_e > D_1$, the sign of this term depends on $\underline{\theta} - 2D_1 + \sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}$. When $2D_1 < \underline{\theta}$, this term is of course larger than zero. When $2D_1 > \underline{\theta}$, it can be verified that $(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta} > (2D_1 - \underline{\theta})^2$. Again, we have the term is larger than zero. When $\underline{\theta} > 0$, we proved that

$$\frac{\partial}{\partial \underline{\theta}}\left[D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right] > 0.$$

Notice that $D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1) = 0$ when $\underline{\theta} = 0$. As a result, when $\underline{\theta} > 0$, this term is larger than zero. In the end, we have

$$sgn\left(\frac{\partial IL}{\partial \Delta}\right) = -sgn\left(D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right) < 0.$$

Increasing capital reduces illiquidity risk in the cases where $\underline{\theta} > 0$.

To summarize, when $\underline{\theta} = 0$, increasing capital has no effect on illiquidity risk. When $\underline{\theta} > 0$, Increasing capital reduces illiquidity risk. But one thing should be emphasized is that increasing capital is less able to reduce illiquidity because of the "inferencing effect". □

Appendix B.3 Lemma 2. The monotonicity of $F_1(\theta)$

Proof. We start with $F_1(\theta)$, the buyers' expected payoff when they expecting one bank run. With the ex post beliefs about state estabilished, $F_1(\theta)$ can be explicitly expressed as:

$$\begin{aligned} F_1(\theta) &= \frac{\theta - \underline{\theta}_B}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)}\frac{\theta + \underline{\theta}_B}{2} + \frac{\theta - \underline{\theta}_G}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)}\frac{\theta + \underline{\theta}_G}{2} - \frac{qD_1}{1 - \frac{D_2 - D_1}{\theta}} \\ &= \frac{1}{2}\frac{2\theta^2 - (\underline{\theta}_B^2 + \underline{\theta}_G^2)}{2\theta - (\underline{\theta}_B + \underline{\theta}_G)} - \frac{qD_1\theta}{\theta - (D_2 - D_1)} \end{aligned}$$

To check the monotonicity of $F_1(\theta)$, we take the derivative:

$$\frac{dF_1(\theta)}{d\theta} = \frac{1}{2}\frac{[2\theta - (\underline{\theta}_B + \underline{\theta}_G)]^2 + (\underline{\theta}_B - \underline{\theta}_G)^2}{[2\theta - (\underline{\theta}_B + \underline{\theta}_G)]^2} + \frac{qD_1(D_2 - D_1)}{[\theta - (D_2 - D_1)]^2} > 0$$

□

Appendix B.4 Proposition 4. The existence and uniqueness of θ_1^*

Proof. It takes two steps to prove Proposition 4. First, we prove the existence and the uniqueness of θ_1^* in the interval $[D_2, \overline{\theta}]$. Second, we prove the equilibrium cash flow $\theta_1^* \in [\theta^L, \theta^U(P_1^*)]$ and the equilibrium price $P_1^* \in (D_1, D_2)$. Note that $F_1(\theta)$ can be rewritten as

$$F_1(\theta) = \omega_1^B(\theta, 1)\pi^B(\theta) + \omega_1^G(\theta, 1)\pi^G(\theta),$$

where $\pi^s(\theta) = \frac{\underline{\theta}_s + \theta}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}$. Thus, the equilibrium condition can be also rewritten as

$$\omega_1^B(\theta_1^*)\pi^B(\theta_1^*) + \omega_1^G(\theta_1^*)\pi^G(\theta_1^*) = 0 \tag{B.27}$$

Step 1: We prove by continuity that there exists $\theta_1^* \in [D_2, \overline{\theta}]$ such that $F_1(\theta_1^*) = 0$.

We value the function $F_1(\theta)$ at $\theta = D_2$. Notice that

$$\omega_1^B(D_2) = \frac{D_2 - \underline{\theta}_B}{(D_2 - \underline{\theta}_B) + (D_2 - \underline{\theta}_G)} > 0 \quad \text{and} \quad \omega_1^G(D_2) = \frac{D_2 - \underline{\theta}_G}{(D_2 - \underline{\theta}_B) + (D_2 - \underline{\theta}_G)} > 0.$$

Moreover, as q sufficiently close to 1, it holds that

$$\pi^B(D_2) = \frac{D_2 + \underline{\theta}_B}{2} - qD_2 < 0 \quad \text{and} \quad \pi^G(D_2) = \frac{D_2 + \underline{\theta}_G}{2} - qD_2 < 0,$$

by the parameter assumption 1. Therefore, we have $F_1(D_2) < 0$.

Now we examine $F_1(\theta)$ at $\theta = \overline{\theta}$. Similarly, we have

$$\omega_1^B(\overline{\theta}) = \frac{\overline{\theta} - \underline{\theta}_B}{(\overline{\theta} - \underline{\theta}_B) + (\overline{\theta} - \underline{\theta}_G)} > 0, \quad \text{and} \quad \omega_1^G(\overline{\theta}) = \frac{\overline{\theta} - \underline{\theta}_G}{(\overline{\theta} - \underline{\theta}_B) + (\overline{\theta} - \underline{\theta}_G)} > 0.$$

And under our assumption 2, it holds that

$$\pi^B(\overline{\theta}) = \frac{\overline{\theta} + \underline{\theta}_B}{2} - \frac{qD_1\overline{\theta}}{\overline{\theta} - (D_2 - D_1)} > 0 \quad \text{and} \quad \pi^G(\overline{\theta}) = \frac{\overline{\theta} + \underline{\theta}_G}{2} - \frac{qD_1\overline{\theta}}{\overline{\theta} - (D_2 - D_1)} > 0.$$

These inequalities hold when q is sufficiently close to 1 as

$$\lim_{q\to 1^-} \pi^B(\overline{\theta}) = \frac{(\overline{\theta}+\underline{\theta}_B)(\overline{\theta}-F)-2D_1\overline{\theta}}{\overline{\theta}-F} > \frac{2D_2(\overline{\theta}-F)-2D_1\overline{\theta}}{\overline{\theta}-F} > \frac{2F(\overline{\theta}-D_2)}{\overline{\theta}-F}$$

Notice that D_2-D_1 tends to F when $q \to 1$[1]. The first inequality is by the efficiency assumption 2, $\frac{\overline{\theta}+\underline{\theta}_B}{2} > D_2$. And $\overline{\theta} > D_2$ follows the efficiency assumption 2 as well. The proof $\pi^G(\overline{\theta}) > 0$ of course holds. Therefore, we have: $F_1(\overline{\theta}) > 0$. By the continuity of function $F_1(\theta)$, there exists $\theta_1^* \in (D_2, \overline{\theta})$ such that $F_1(\theta_1^*) = 0$.

Step 2: We prove $P_1^* \in (D_1, D_2)$ and $\theta_1^* < \theta^U(P_1^*)$.

Note that $\theta_1^* = \frac{D_2-D_1}{1-qD_1/P_1^*}$ holds in equilibrium. This P_1^* is unique for $\theta_1^* \in [D_2, \overline{\theta}]$, as such θ_1^* is unique. Moreover, P_1^* can not be equal to D_1. Otherwise, θ_1^* can never belong to a finite region $[D_2, \overline{\theta}]$.

We then consider the case when $q \to 1$: $\lim_{q\to 1} \frac{D_2-D_1}{1-\frac{q}{P_1^*}D_1} = \frac{D_2-D_1}{1-\frac{D_1}{P_1^*}}$. It can be seen that $\frac{D_2-D_1}{1-\frac{D_1}{P_1^*}} > 0$ only if $P_1^* > D_1$ and $\frac{D_2-D_1}{1-\frac{D_1}{P_1^*}} > \theta^L = D_2$ only if $P_1^* < D_2$. Thus, we prove P_1^* belongs to (D_1, D_2). With $\theta_1^* = \frac{D_2-D_1}{1-qD_1/P_e^*}$, $F_1(\theta_1^*) = 0$ can be also rewritten as $\omega_1^B(P_1^*)\cdot\pi^B(P_1^*) + \omega_1^G(P_1^*)\cdot\pi^G(P_1^*) = 0$. Hence, such P_1^* indeed makes the asset buyers earn zero profit when one bank run is observed.

Similar as in Appendix B.1, the derivative $lim_{q\to 1^-}\frac{d}{dq}\left[\theta^U(P_1^*) - \hat{\theta}(P_1^*)\right] = \frac{P_1^*-D_2}{P_1^*}$. Having proved $P_1^* < D_2$, $\theta^U(P_1^*) > \hat{\theta}(P_1^*)$ when $q \in (1-\epsilon, 1)$. That is $\theta_1^* \in [D_2, \theta^U(P_1^*)] \subset [D_2, \overline{\theta}]$. Recall Appendix A, such $\theta_1^* = \frac{D_2-D_1}{1-qD_1/P_1^*}$ is indeed a threshold equilibrium given asset price P_1^*.

To summarize, the unique price $P_1^* \in (D_1, D_2)$ indeed makes the asset buyers make zero profit, and no incentive to deviate. And the unique $\theta_1^* \in [D_2, \overline{\theta}]$ is indeed an equilibrium critical cash flow. Combine those two, the equilibrium $\{\theta_1^*, P_1^*\}$ exists and is unique when one bank run is observed. □

Appendix B.5 Lemma 3. The monotonicity of $F_2(\theta)$

Proof. We show the monotonicity of $F_2(\theta)$, the buyers' expected payoff when they expecting two bank runs. We write explicitly function $F_2(\theta)$ as:

$$F_2(\theta) = \frac{1}{2}\left[\frac{(\theta-\underline{\theta}_B)^2(\theta+\underline{\theta}_B)}{(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2} + \frac{(\theta-\underline{\theta}_G)^2(\theta+\underline{\theta}_G)}{(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2}\right] - \frac{qD_1\theta}{\theta-(D_2-D_1)}$$

Again, we take the derivative of $F_2(\theta)$ respect to θ. The derivative to θ of the first term in the parenthesis is:

$$\frac{2(\theta+\underline{\theta}_B)[(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)-(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)]+(\theta-\underline{\theta}_B)^4+(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)^2}{[(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2]^2}$$

The derivative to θ of the second term in the parenthesis is:

$$\frac{2(\theta+\underline{\theta}_G)[(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)-(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)]+(\theta-\underline{\theta}_G)^4+(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)^2}{[(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2]^2}$$

Notice that

$$\begin{aligned}&2(\theta+\underline{\theta}_B)[(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)-(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)]+2(\theta+\underline{\theta}_G)[(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)-(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)]\\&=2(\underline{\theta}_G-\underline{\theta}_B)^2(\theta-\underline{\theta}_B)(\theta-\underline{\theta}_G)>0\end{aligned}$$

And the derivative for the last term is again, $-\frac{dP(\theta)}{d\theta}$, positive. Put these discussions altogether, we obtain

$$\frac{dF_2(\theta)}{d\theta}=\frac{2(\underline{\theta}_G-\underline{\theta}_B)^2(\theta-\underline{\theta}_B)(\theta-\underline{\theta}_G)+(\theta-\underline{\theta}_B)^4+(\theta-\underline{\theta}_G)^4+2(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)^2}{2[(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2]^2}+\frac{qD_1(D_2-D_1)}{[\theta-(D_2-D_1)]^2}>0$$

□

Appendix B.6 Proposition 5. The existence and uniqueness of θ_2^*

Proof. We follow the same argument as the proof in Appendix B.4. Similar;y, the equilibrium condition can be expressed as

$$F_2(\theta_2^*)=\omega_2^B(\theta_2^*)\pi^B(\theta_2^*)+\omega_2^G(\theta_2^*)\pi^G(\theta_2^*)=0 \tag{B.28}$$

To check the step 1. Notice that:

$$\omega_2^B(D_2)=\frac{(D_2-\underline{\theta}_1)^2}{(D_2-\underline{\theta}_B)^2+(D_2-\underline{\theta}_G)^2}>0,\qquad \omega_2^G(D_2)=\frac{(D_2-\underline{\theta}_G)^2}{(D_2-\underline{\theta}_B)^2+(D_2-\underline{\theta}_G)^2}>0.$$

Moreover,

$$\omega_2^B(\overline{\theta})=\frac{(\overline{\theta}-\underline{\theta}_B)^2}{(\overline{\theta}-\underline{\theta}_B)^2+(\overline{\theta}-\underline{\theta}_G)^2}>0,\qquad \omega_2^G(\overline{\theta})=\frac{(\overline{\theta}-\underline{\theta}_G)^2}{(\overline{\theta}-\underline{\theta}_B)^2+(\overline{\theta}-\underline{\theta}_G)^2}>0$$

The sign of function $F_2(\theta)$ depends on $\pi^B(\theta)$ and $\pi^G(\theta)$, which have the same definitions as in Appendix B.4. We have already showed that: $\pi^B(D_2) < 0, \pi^B(\overline{\theta}) > 0$ and $\pi^G(D_2) < 0, \pi^G(\overline{\theta}) > 0$. Thus we can again claim:

$$F_2(D_2) < 0 \qquad \text{and} \qquad F_2(\overline{\theta}) > 0.$$

By the continuity of $F_2(\theta)$, there exists a $\theta_2^* \in (D_2, \overline{\theta})$ satisfying $F_2(\theta_2^*) = 0$. Then by Lemma 1, θ_2^* necessarily belongs to $\left(D_2, \theta^U(P_2^*)\right)$ with $P_2^* = \frac{qD_1\theta_2^*}{\overline{\theta}-(D_2-D_1)}$.

Since F_2 is monotonically increasing in θ, the uniqueness of this θ_2^* is again guaranteed. The equilibrium $\{\theta_2^*, P_2^*\}$ exists and is unique.

Then step 2 follows exactly the procedure as in Appendix B.4, we thus omit it. □

Appendix B.7 Proposition 6. Financial contagion

Proof. The proof hinges on the monotonicity of two ratios

$$\frac{\omega_2^B(\theta)}{\omega_2^G(\theta)} = \frac{(\theta - \underline{\theta}_B)^2}{(\theta - \underline{\theta}_G)^2} \qquad \text{and} \qquad \frac{\pi^G(\theta)}{\pi^B(\theta)} = \frac{\frac{\theta+\underline{\theta}_G}{2} - P(\theta)}{\frac{\theta+\underline{\theta}_B}{2} - P(\theta)}.$$

The first is a conditional likelihood ratio and the second is a payoff ratio. It can be shown both ratios are strictly monotonically decreasing in θ, that is

$$\frac{d}{d\theta}\left(\frac{\omega_2^B(\theta)}{\omega_2^G(\theta)}\right) = -\frac{2(\theta - \underline{\theta}_B)(\underline{\theta}_G - \underline{\theta}_B)}{(\theta - \underline{\theta}_G)} < 0$$

$$\frac{d}{d\theta}\left(\frac{\pi^G(\theta)}{\pi^B(\theta)}\right) = -\frac{[\frac{1}{2} - P'(\theta)][\frac{\underline{\theta}_G-\underline{\theta}_B}{2}]}{[\frac{\theta+\underline{\theta}_B}{2} - P(\theta)]} < 0$$

We focus on the interior realization of cash flow, then $\theta > \underline{\theta}_G$. And remember $P'(\theta) < 0$ from the Appendix B.4.

Furthermore, notice that for $\omega_1^B(\theta)/\omega_1^G(\theta) > 1$, we have

$$\frac{\omega_1^B(\theta)}{\omega_1^G(\theta)} < \left[\frac{\omega_1^B(\theta)}{\omega_G(\theta)}\right]^2 = \frac{\omega_2^B(\theta)}{\omega_2^G(\theta)} \tag{B.29}$$

Now we prove by contradiction. Suppose $\theta_1^* > \theta_2^{**}$. By the monotonicity of $\pi^G(\theta)/\pi^B(\theta)$, we will have

$$\frac{\pi^G(\theta_1^*)}{\pi^B(\theta_1^*)} < \frac{\pi^G(\theta_2^{**})}{\pi^B(\theta_2^{**})}.$$

By the equilibrium conditions (B.27) and B.28, we have

$$\frac{\pi^G(\theta_1^*)}{\pi^B(\theta_1^*)} = -\frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} \qquad \text{and} \qquad \frac{\pi^G(\theta_2^{**})}{\pi^B(\theta_2^{**})} = -\frac{\omega_2^B(\theta_2^{**})}{\omega_2^G(\theta_2^{**})},$$

which implies

$$\frac{\omega_2^B(\theta_2^{**})}{\omega_2^G(\theta_2^{**})} < \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)}.$$

By condition (B.29), we know

$$\frac{\omega_2^B(\theta_2^{**})}{\omega_2^G(\theta_2^{**})} < \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} < \frac{\omega_2^B(\theta_1^*)}{\omega_2^G(\theta_1^*)}.$$

But this contradicts the monotonicity of $\omega_2^B(\theta)/\omega_2^G(\theta)$. Therefore, we prove $\theta_2^{**} > \theta_1^*$. □

Appendix B.8 Lemma 4. Regulator's break even price P_A^*

Proof. By inserting (21) into (22), one can obtain the following equation.

$$4(P_A)^2 - [2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)]P_A + qD_1(\underline{\theta}_B + \underline{\theta}_G) = 0$$

The positive solution of this quadratic function is

$$P_A^* = \frac{[2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)] + \sqrt{[2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)]^2 - 16qD_1(\underline{\theta}_B + \underline{\theta}_G)}}{8}$$

Following the proof in Appendix B.1, we can check that $P_A^* \in (D_1, D_2)$. Moreover, we can also check that the regulator does not have profitable deviation by unilaterally bid higher price than P_A^*. □

Appendix B.9 Proposition 7. Asset purchase

Proof. Recall that θ_1^* solves $F_1(\theta_1^*) = 0$. $F_1(\theta)$ can be rewritten as

$$F_1(\theta) = \frac{1}{2}\frac{\underline{\theta}_B + \theta}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \theta}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)} - \frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4[(\theta - \underline{\theta}_G) + (\theta - \underline{\theta}_B)]}$$

While, we can define

$$F_A(\theta) = \frac{1}{2}\frac{\underline{\theta}_B + \theta}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \theta}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)},$$

where $F_A(\theta_A^*) = 0$. Insert θ^a into it, we have

$$F_1(\theta_A^*) = F_A(\theta_A^*) - \frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4[(\theta - \underline{\theta}_G) + (\theta - \underline{\theta}_B)]} = -\frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4[(\theta - \underline{\theta}_G) + (\theta - \underline{\theta}_B)]} < 0$$

Recall again $F_1(\theta)$ is increasing in θ. We have $\theta_A^* < \theta_1^*$. Then $P_A^* > P_1^*$ immediately follows. □

Appendix B.10 Lemma 5. Regulatory disclosure and bank runs

Proof. We solve here only for the equilibrium in state $s = G$. The equilibrium under $s = 1$ can be solved with the same procedure. The equilibrium is determined by a system of two equations:

$$\begin{cases} \theta_e^G = \frac{D_2 - D_1}{1 - \frac{q}{P_e^G} D_1} \\ P_e^G = \frac{\theta_e^G + \underline{\theta}_G}{2} \end{cases}$$

Solving the system of equations as in the Appendix B, we have the equilibrium critical cash flow and the endogenous fire-sale price:

$$\theta_e^G = \frac{(D_2 - D_1) + 2qD_1 - \underline{\theta}_G + \sqrt{[(D_2 - D_1) + 2qD_1 - \underline{\theta}_G]^2 + 4(D_2 - D_1)\underline{\theta}_G}}{2}$$
$$P_e^G = \frac{(D_2 - D_1) + 2qD_1 + \underline{\theta}_G \pm \sqrt{[(D_2 - D_1) + 2qD_1 + \underline{\theta}_G]^2 - 8qD_1\underline{\theta}_G}}{4}$$

When q is sufficiently close to 1, we have

$$\theta_e^G = \frac{(D_1 + D_2 - \underline{\theta}_G) + \sqrt{[D_1 + D_2 - \underline{\theta}_G]^2 + 4F\underline{\theta}_G}}{2}$$
$$P_e^G = \frac{(D_1 + D_2 + \underline{\theta}_G) + \sqrt{[D_1 + D_2 + \underline{\theta}_G]^2 - 8D_1\underline{\theta}_G}}{4}$$

It is straightforward to check that $\theta_G^* \in (D_2, \underline{\theta}]$ as in Appendix B.2. □

Appendix B.11 Proposition 8. Regulatory disclosure and illiquidity

Proof. We start by proving $\theta_e^G < \theta_1^*$. Recall that $F_1(\theta_1^*) = 0$ and $F_1(\theta)$ is monotonically increasing. So $\theta_e^G < \theta_1^*$ will hold if and only if $F_1(\theta_e^G) < 0$. To proceed, we write $F_1(\theta)$ explicitly

$$F_1(\theta) = \frac{\theta - \underline{\theta}_B}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\theta + \underline{\theta}_B}{2} + \frac{\theta - \underline{\theta}_G}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\theta + \underline{\theta}_G}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}.$$

We can rewrite $F_1(\theta)$ as follows

$$\begin{aligned} F_1(\theta) &= \frac{\theta-\underline{\theta}_B}{(\theta-\underline{\theta}_B)+(\theta-\underline{\theta}_G)}\left[\frac{\theta+\underline{\theta}_B}{2}-\frac{\theta+\underline{\theta}_G}{2}\right]+\frac{\theta+\underline{\theta}_G}{2}-\frac{qD_1\theta}{\theta-(D_2-D_1)}. \\ &= -\frac{\theta-\underline{\theta}_B}{(\theta-\underline{\theta}_B)+(\theta-\underline{\theta}_G)}\frac{\underline{\theta}_G-\underline{\theta}_B}{2}+\frac{\theta+\underline{\theta}_G}{2}-\frac{qD_1\theta}{\theta-(D_2-D_1)}. \end{aligned}$$

We then evaluation $F_1(\theta)$ at θ_e^G, that is

$$F_1(\theta_e^G) = -\frac{\theta_e^G-\underline{\theta}_B}{(\theta_e^G-\underline{\theta}_B)+(\theta_e^G-\underline{\theta}_G)}\frac{\underline{\theta}_G-\underline{\theta}_B}{2} < 0.$$

Remember that the term $\frac{\theta_e^G+\underline{\theta}_G}{2}-\frac{qD_1\theta_e^G}{\theta_e^G-(D_2-D_1)} = \frac{\theta_e^G+\underline{\theta}_G}{2}-P_e^G = 0$. Then we have $\theta_e^G < \theta_e^*$.

We then prove $\theta_e^B > \theta_2^{**}$. Recall that $F_2(\theta_e^{**}) = 0$ and $F_2(\theta)$ is monotonically increasing. So $\theta_e^B > \theta_e^{**}$ will hold if and only if $F_2(\theta_e^B) > 0$. Similarly, we can write $F_2(\theta)$ as

$$F_2(\theta) = \frac{(\theta-\underline{\theta}_G)^2}{(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2}\frac{\underline{\theta}_G-\underline{\theta}_B}{2}+\frac{\theta+\underline{\theta}_B}{2}-\frac{qD_1\theta}{\theta-(D_2-D_1)}.$$

We evaluation $F_2(\theta)$ at θ_e^B, and for the similar argument

$$F_2(\theta_e^B) = \frac{(\theta_e^B-\underline{\theta}_G)^2}{(\theta_e^B-\underline{\theta}_B)^2+(\theta_e^B-\underline{\theta}_G)^2}\frac{\underline{\theta}_G-\underline{\theta}_B}{2} > 0$$

Then we have $\theta_e^B > \theta_2^{**}$.

□

Appendix B.12 Proposition 9. Socially undesirable disclosure

Proof. It can be seen easily $SC^{AP} < SC^{RT}$ if and only if $\theta^a < \frac{\theta_e^G+\theta_e^B}{2}$. Consider the auxiliary function

$$G(\theta) = \frac{2qD_1\theta}{\theta-(D_2-D_1)} - \theta.$$

Then θ^a satisfies $G(\theta^a) = \frac{\underline{\theta}_B+\underline{\theta}_G}{2}$. θ_e^G and θ_e^B combined satisfy $\frac{1}{2}G(\theta_e^G)+\frac{1}{2}G(\theta_e^B) = \frac{\underline{\theta}_B+\underline{\theta}_G}{2}$. Together we obtain

$$G(\theta^a) = \frac{1}{2}G(\theta_e^G)+\frac{1}{2}G(\theta_e^B)$$

It is fairly easy to check that $G' = -\frac{2qD_1(D_2-D_1)}{[\theta-(D_2-D_1)]^2} < 0$ and $G''(\theta) = \frac{4qD_1(D_2-D_1)[\theta-(D_2-D_1)]}{[\theta-(D_2-D_1)]^4} > 0$, thus G is a decreasing convex function. We further have

$$G(\theta^a) = \frac{1}{2}G(\theta_e^G) + \frac{1}{2}G(\theta_e^B) > G(\frac{\theta_e^G + \theta_e^B}{2})$$

Lastly, because the function G is decreasing, we obtain $\theta_A^* < \frac{\theta_e^G+\theta_e^B}{2}$. The social cost due to illiquidity is lower than the regulator chooses to implement the asset purchase program.

□