Contagious Bank Runs and Committed Liquidity Support*

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Abstract

In a crisis, regulators and private investors can find it difficult, if not impossible, to tell whether banks facing runs are insolvent or merely illiquid. We introduce such an information constraint into a global-games-based bank run model with multiple banks and aggregate uncertainties. The information constraint creates a vicious cycle between contagious bank runs and falling asset prices and limits the effectiveness of traditional emergency liquidity assistance programs. We explain how a regulator can set up committed liquidity support to contain contagion and stabilize asset prices even without information on banks' solvency, rationalizing some recent developments in policy practices.

Keywords: Committed liquidity support, Global games, Bank runs

JEL Classification: G01, G11, G21

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1 Introduction

The 2007-2009 financial crisis highlights a dual-illiquidity problem. During the crisis, market liquidity evaporated, and asset prices dropped sharply. At the same time, funding liquidity dried up, and even well-capitalized banks found it difficult to roll over their short-term debt. In response to the dual-illiquidity problem, central banks were creative in providing facilities for liquidity support during the crisis and have been experimenting with novel upfront liquidity arrangements afterward. Examples of such upfront liquidity arrangements include the asset pre-positioning program of the Bank of England and the committed liquidity facility of the Reserve Bank of Australia. In this paper, we model the two-way feedback between distressed asset prices and contagious bank runs and show how upfront liquidity support can be an effective response to the dual-illiquidity problem, rationalizing the recent policy developments.

Central to our model is the observation that it can be difficult — if not impossible — to distinguish illiquid banks from insolvent ones in crisis times.[1] We show that such an information constraint creates a vicious cycle between falling asset prices and bank runs. When private asset buyers cannot distinguish assets sold by illiquid banks from those sold by insolvent banks, the price they offer would reflect the average asset quality between these two. As a result, an illiquid bank would be unable to recoup a fair value for its assets on sale. In a global-games framework, we show that creditors' expectations of low asset prices due to this information friction can deprive a solvent bank of its short-term funding. Each creditor, anticipating the liquidation loss caused by other creditors' early withdrawals from his bank, chooses to join the run himself. However, it is the run and the forced liquidation — by pooling the illiquid bank with insolvent ones — that lead to the decline in asset prices in the first place.[2]

In a two-bank setting, financial contagion and systemic crisis emerge once we introduce an aggregate risk factor that affects both banks' fundamentals. We analyze a global game with multiple groups of players (i.e., two distinct groups of creditors for the two banks) and multi-dimensional signals (i.e., in addition to the private signals about their own bank's fundamentals, creditors receiving a common signal about the other bank's fundamentals as outsiders). We show that coordination failures occur not only among creditors within a bank but also between creditors from different banks. The cross-bank coordination failure is triggered by the expecta-

[1] The information constraint is recognized as one of the main challenges for central banks to act as lenders of last resort. See e.g., Goodhart (1999) and Freixas et al. (2004).

[2] Using historical data, Fohlin et al. (2016) empirically document the feedback between market and funding illiquidity, providing evidence that information asymmetry on asset qualities contributes to the vicious cycle.

tion of falling asset prices: upon observing more bank runs, the asset buyers' beliefs about the aggregate risk deteriorate, which reduces their bids for banks' assets. The lower asset prices, in turn, precipitate runs at more banks. The cross-bank coordination failure suggests financial contagion: a bank is more likely to experience runs when its creditors perceive the other bank to have weak fundamentals and expect that bank's liquidation to depress asset prices. We derive a unique equilibrium despite the two-way feedback between collapsing asset prices and contagious bank runs.

The information constraint that creates the financial fragility also limits the effectiveness of traditional emergency liquidity assistance programs. An informationally constrained central bank cannot lend only to the solvent-but-illiquid banks suggested by the classic lender-of-last-resort (LoLR) principles of Bagehot (1873). In particular, in tackling banks' funding problems as runs happen, an informationally constrained central bank risks rescuing insolvent banks or making losses from the intervention.

We show that upfront liquidity support can contain contagious bank runs even if the central bank holds no information on individual banks' solvency. Intuitively, a central bank can support the price of banks' assets in a pre-committed arrangement, thereby breaking down the two-way feedback between falling asset prices and contagious bank runs. We recommend an arrangement where a regulator and banks mutually commit to an agreement for the regulator to purchase a bank's assets for a pre-specified price when the bank experiences runs.[3] In making her offer before the aggregate risk is realized, the regulator's price support is neither conditional on the aggregate state nor on the knowledge about the banks' solvency. The pre-specified price allows the regulator to contain the risk of contagion while breaking even across possible posterior beliefs about the aggregate risk from an ex-ante perspective.

Our modeling of the committed liquidity support is broadly consistent with the suggestion of King (2017) that a central bank should act as a "pawnbroker for all seasons (PFAS)" and commits to providing liquidity insurance to banks in times of crisis. Our theory suggests that the liquidity support is the most effective if banks commit to raising liquidity from the central bank when experiencing runs. This suggested regulatory obligation is also in line with King's proposals that, for emergency liquidity assistance, banks should be "required to take out insurance in the form of pre-positioned collateral with the central bank" and that the provision of liquidity insurance should be "mandatory and paid for upfront".

[3] While an arrangement with the regulator's unilateral commitment can also promote stability, as we will show in Section 4.2, an arrangement with mutual commitments can delivery the same stability effect with a lower cost.

The recent policy practices have indeed seen the implementations of such committed liquidity support. The Bank of England has implemented asset pre-positioning as a part of its Sterling Monetary Framework.[4] Also in the spirit of the proposal in King (2017), the Reserve Bank of Australia launched the committed liquidity facility with an explicit requirement for banks to commit to the regime. To benefit from the liquidity support of the facility, a bank needs to pay a premium of 15 basis points for the amount of liquidity committed by the central bank. In return, the central bank contractually commits to entering repo transactions with the participating bank, should runs happen to it.[5]

This paper makes three contributions. First, we introduce a relevant information constraint into a global-games-based bank run model: it is difficult to distinguish illiquid banks from insolvent ones during crisis times. We show that the information constraint not only results in a vicious cycle between bank runs and distressed asset prices, but also makes traditional policy interventions ineffective. Second, we analyze a novel global-games setting with multiple groups of players and multi-dimensional signals. In addition to the coordination problem among creditors within a bank, the unique equilibrium of the game also features strategic complementarities between creditors from different banks, which captures financial contagion. Finally, from a policy perspective, we show that an effective liquidity intervention features committed liquidity support, providing a formal theory to interpret some of the recent developments in central bank policy practices.

Related literature: Our paper contributes to the literature on public liquidity intervention and global-games-based bank run models. Central bank liquidity injection in a global-games framework was first studied by Rochet and Vives (2004). The authors consider a single-bank setup and derive a unique threshold equilibrium that features solvent-but-illiquid banks as in Bagehot (1873). The authors further assume that banks' fundamentals are perfectly observable to the central bank and suggest that the central bank can act as a LoLR by lending directly and only to solvent banks. In a two-bank setting, we generalize Rochet and Vives (2004) by introducing information constraints, endogenous liquidation value, and aggregate uncertainty. We focus on systemic crises instead of runs on individual banks and show that an upfront liquidity insurance policy can mitigate system-wide financial fragility.

[4] By the spring of 2015, £469 billions of bank assets had been pre-positioned with the central bank, with an average haircut of 33%. In January 2019, the central bank published detailed guidelines regarding the procedures for private institutions to pre-position their illiquid assets. See Bank of England (2019).

[5] See Reserve Bank of Australia (2018, 2019) for details. By the end of 2018, a total of AU$ 248 billions of central bank liquidity support was committed through the facility to eligible banks.

Our model predicts a vicious cycle between bank runs and falling asset prices, and strategic complementarities between creditors from different banks. These two features are most related to Liu (2016) and Goldstein et al. (2020), respectively. Liu (2016) studies how limited participation in the interbank market can lead to an interaction between bank runs and rising interbank market rates.[6] Goldstein et al. (2020) allow for cross-bank coordination failures to study the impact of bank heterogeneity on financial stability.[7] Both papers, however, assume that the (external) providers of liquidity — i.e., the lending banks in Liu (2016) and the asset buyers in Goldstein et al. (2020) — can perfectly observe the fundamentals of the bank that is experiencing runs. By contrast, we emphasize that both central banks and private investors can face the information constraint in telling whether runs happen to a bank due to its insolvency or mere illiquidity. We emphasize that central banks' liquidity support should be designed with the information constraint taken into account. The committed liquidity support that we propose rationalizes some of the recent developments in central bank practices.[8]

Our paper also contributes to the debate on central banks' role as lenders of last resort. Goodfriend and King (1988) and Freixas et al. (2004) argue that when it is hard to tell whether an illiquid bank is solvent, it can be optimal for central banks to only provide liquidity in open market operations and let the interbank market allocate the liquidity. The others question the view, arguing that the asymmetric information about banks' solvency can also interrupt the functioning of the interbank market and justify central banks' direct lending to banks (Flannery (1996), Heider et al. (2015), and Choi et al. (2017)). We study a setting where neither central banks nor the private investors possess precise information on the solvency of the illiquid bank and suggest that setting up upfront liquidity support can be more effective than providing ex-post emergency liquidity assistance as runs happen. We are also the first to study this information constraint in a global-games framework, which should be a natural setting since the framework endogenously defines solvent-but-illiquid banks.

[6]In a non-bank setup without coordination failures, Brunnermeier and Pedersen (2009) highlights two-way feedback between market and funding illiquidity by emphasizing a margin constraint on a speculator who supplies liquidity to a financial market with limited participation. In their model, asset prices are volatile because the selling and buying of assets are not synchronized. By contrast, we emphasize the funding liquidity risk caused by equilibrium bank runs and that asymmetric information on asset qualities causes asset illiquidity.

[7]In addition, Goldstein (2005) and Leonello (2018) also feature cross-entity coordination failures. Our approach differs from all the three papers as we solve a model with multiple groups of players, each of whom receives multi-dimensional signals. Different from the focus of the current paper, Goldstein (2005) and Leonello (2018) examine how bank runs interact with currency crisis and sovereign bond crisis, respectively.

[8]Liu (2016) also discusses a policy intervention, which is modeled as an ex-post net transfer from the central bank to private institutions, conditional on the central bank's observation of bad state. In contrast, we emphasize that the liquidity intervention should be pre-emptive: it should be announced before the realization of the aggregate uncertainty and can still be effective even if the central bank does not observe the aggregate state.

In terms of analyzing global games with multi-dimensional signals, our paper is most related to Fujimoto (2014), who studies a game where one group of speculators learn private signals about multiple countries and choose only one country's currency to attack.[9] Assuming a short-selling constraint for the speculators, Fujimoto (2014) shows that the speculators' attack at one country makes them less likely to attack the other countries. By contrast, the multi-dimensional signals in our model with two groups of players generate strategic complementarities among creditors from different banks.

The paper proceeds as follows. Section 2 lays out our model. Section 3 characterizes the model's equilibrium, showing how a vicious cycle between falling asset prices and contagious bank runs can emerge in a laissez-faire market. We then show how committed liquidity support can mitigate such financial fragility. We extend the policy discussion in Section 4 and conclude in Section 5.

2 Model setup

We consider a three-date ($t = 0, 1, 2$) economy with two banks ($i = 1, 2$). There are two types of risk-neutral players: banks' wholesale creditors and secondary-market asset buyers.

2.1 Banks

The two banks are identical at $t = 0$. Each of them holds a unit portfolio of long-term assets and finances the portfolio with equity E, retail deposits F, and short-term wholesale debt $1 - E - F$. We consider banks as contractual arrangements among claim holders and designed to fulfill the function of liquidity transformation. Therefore, banks in our model are passive, with given loan portfolios and liability structures.

A bank i's assets generate a random cash flow $\tilde{\theta}^i \sim U(\underline{\theta}_s, \overline{\theta})$. The realization of the cash flow is not only affected by the idiosyncratic risk of the bank, but also by a systematic risk factor s. The systematic risk, as indicated by the subscript of the lower bound, affects the distribution of both banks' cash flows. There are two possible aggregate states, $s = G$ and $s = B$. With $\underline{\theta}_G > \underline{\theta}_B > 0$, the distribution of the cash flows in state G first order stochastic dominates that in state B. All players hold a prior belief that State G and B occur with probabilities α

[9] In their seminal work, Carlsson and Van Damme (1993) allow the state variables that determine players' payoffs to be multidimensional and players only observe noisy signals about the multidimensional state variables. Oury (2013) analyzes when the unique equilibrium selection does not depend on the distribution of the noises about the multi-dimensional state variables.

and $1-\alpha$, respectively. Note that the upper bound of the cash flows remains the same across the two aggregate states, reflecting the fact that banks hold mostly debt claims whose highest payoffs are capped by their face values. Once the aggregate state s is realized, the two banks' cash flows are determined by their idiosyncratic risks and assumed to be independently and identically distributed. The realized fundamentals of the banking sector can be represented by a vector $(\theta^1, \theta^2) \in [\underline{\theta}_s, \overline{\theta}] \times [\underline{\theta}_s, \overline{\theta}]$.

On the liability side, we assume that a bank is financed by retail depositors and its pool of wholesale creditors. The retail depositors are fully protected by deposit insurance, which is provided to the bank free of charge. Therefore, retail depositors passively hold their claims to maturity and demand only a gross risk-free rate that we normalize to 1. On the other hand, each bank's wholesale debt is risky, demandable, and raised from a distinct continuum of creditors of mass 1. Provided that the bank does not fail, the wholesale debt pays a gross interest rate $r_D > 1$ if a wholesale creditor waits till $t = 2$, and qr_D if the wholesale creditor withdraws early at $t = 1$. Here, $q < 1$ reflects the penalty for the early withdrawal.[10] For the ease of presentation, we denote by $D_1 \equiv (1 - E - F)qr_D$ the total amount of debt that a bank needs to repay at $t = 1$ if *all* of its wholesale creditors withdraw early, and by $D_2 \equiv (1 - E - F)r_D + F$ the total amount of debt that a bank needs to repay at $t = 2$ if *none* of its wholesale creditors withdraws early. We make the following three parametric assumptions.

$$D_2 > \underline{\theta}_s \tag{1}$$

$$F > D_1 \tag{2}$$

$$q > \frac{1}{2} + \frac{\underline{\theta}_G}{2D_2} \tag{3}$$

Inequality (1) states that banks are not risk-free and face a positive probability of bankruptcy even in State G. Inequality (2) suggests that banks' retail debt exceeds their short-term wholesale debt, which represents a realistic case and helps simplify the analysis of bank run games.[11] Finally, inequality (3) states that the penalty for early withdrawal is only moderate.[12] While we

[10] Jin et al. (2019) and Capponi et al. (2020) consider the risk of runs in a setting of equity mutual funds and point out that the more flexible contract such as swing pricing can mitigate the risk. The current paper focuses on financial institutions with non-contingent liabilities, typically banks with demandable debt financing.

[11] Despite the rapid growth of wholesale funding, most commercial banks and bank holding companies are still financed more by retail deposits than wholesale debt. For example, Cornett et al. (2011) document that the median core deposit to asset ratio for US commercial banks was 67.88% over the period from 2006 to 2009.

[12] For example, when $\underline{\theta}_G = \underline{\theta}_B = 0$, the condition states that $q > 1/2$. That is, by withdrawing early, a wholesale creditor will not lose more than a half of the face value of his claim. The moderate penalty for early withdrawal is in line with banks' role as liquidity providers as suggested by Diamond and Dybvig (1983).

do not endogenize banks' capital structure (thus taking q, D_1, and D_2 as given), as long as the optimal capital structure satisfies the aforementioned conditions, all of our results apply.

2.2 The bank run game

A bank run game of complete information can have multiple equilibria. To refine the multiplicity, we take the global-games approach pioneered by Carlsson and Van Damme (1993) and assume that creditors observe noisy signals of banks' cash flows. We assume that a representative wholesale creditor j of a bank i observes a private signal x_j^i about the realized cash flow of his own bank. Specifically, we assume $x_j^i = \theta^i + \epsilon_j^i$ with the noise ϵ_j^i drawn from a uniform distribution with support $[-\epsilon, \epsilon]$. In addition, all creditors of the bank i observe a signal y^{-i} about the other bank's realized cash flow as outsiders. We assume $y^{-i} = \theta^{-i} + \eta^{-i}$ with the noise η^{-i} drawn from a uniform distribution with support $[-\eta, \eta]$. We assume all noises to be independent. Here, η can be interpreted as a proxy of the banks' transparency to outsiders, with a smaller η associated with greater transparency. We focus on the case $\epsilon < \eta$, so that the wholesale creditors' inside and private signals about their own bank's fundamentals are more accurate than the signal that they receive as outsiders.[13] The information structure presumes players who are more closely linked to a bank receiving more precise signals on the bank's fundamentals, which we consider as a realistic scenario but is not analyzed in the global-games-based bank run literature. The information structure where insiders and outsiders receive different signals is also prevalent in other finance literature, such as papers studying the impact of corporate disclosure, e.g., Goldstein and Yang (2017, 2019) and Yang (2020).

After receiving his signals (x_j^i, y^{-i}), the creditor j from the bank i takes one of two possible actions: to wait till $t = 2$, or to withdraw from his bank at $t = 1$. We focus on the following threshold strategy that is symmetric across all wholesale creditors,[14]

$$(x_j^i, y^{-i}) \longmapsto \begin{cases} \text{withdraw} & x_j^i < x(y^{-i}) \\ \text{wait} & x_j^i \geq x(y^{-i}). \end{cases} \tag{4}$$

That is, a wholesale creditor withdraws from his bank if and only if his private signal about his own bank's fundamentals is below a threshold $x(y^{-i})$. Different from most global-games

[13]This assumption is immaterial. In the Online Appendix, we show that assuming the opposite does not changes the main results of our model.

[14]In the finance application of global games, the threshold equilibrium is of primary interest. For example, see Morris and Shin (2004) and Liu (2016). Since creditors are ex-ante homogenous and banks are also assumed to have *i.i.d.* cash flows and the same capital structure, there is no loss of generality to focus on symmetric strategies.

models, the threshold is not a constant but a function of the signal y^{-i} that the creditor receives as an outsider to the other bank.[15] We assume the function $x(y^{-i})$ to be non-increasing so that the threshold strategy features two types of monotonicity: (1) the creditor's action is monotonic in his inside and private signal, and (2) the threshold is monotonic in the signal that the creditor receives as an outsider. The creditors' equilibrium strategy would be characterized by a function $x^*(\cdot)$. Focusing on $x^*(\cdot)$ that is non-increasing implies that a creditor's incentives to withdraw from his own bank should not be lower when the other bank's fundamentals become weaker. For an equilibrium strategy of creditors', there will be a corresponding critical cash flow of the bank $\theta^*(y^{-i})$ such that the bank fails in a run when its cash flow $\theta^i < \theta^*(y^{-i})$.

A wholesale creditor's payoff depends both on his withdrawal decision and on the bank's solvency. The creditor will receive D_1 if he withdraws early and the bank does not fail at $t = 1$; he will receive D_1/q, if he waits and the bank stays solvent at $t = 2$. In the case of failure, a bank incurs a bankruptcy cost, which can be interpreted as the legal cost of bankruptcy. We assume the cost to be sufficiently high such that if a wholesale creditor waits and the bank fails at either $t = 1$ or $t = 2$, the wholesale creditor will receive a zero payoff and a senior deposit insurance company obtains the residual value of the bank.[16] Finally, we assume that the creditor can obtain an arbitrarily small reputational benefit by running on a bank that fails at $t = 1$.[17]

A wholesale creditor can form rational beliefs about the fraction of withdrawals in his own bank and that in the other bank. We denote the two fractions by L^i and L^{-i}, respectively. We define the occurrence of a bank run as any positive mass of wholesale creditors withdrawing funds from their bank at $t = 1$.[18] By this definition, we have the number of bank runs $M = 1$ when $L^i > 0$ and $L^{-i} = 0$ or $L^i = 0$ and $L^{-i} > 0$. Similarly, $M = 2$ when $L^i > 0$ and $L^{-i} > 0$.

2.3 Secondary asset market

When facing withdrawals at $t = 1$, banks have to liquidate their long-term assets in a secondary asset market. As early liquidation is costly in this model, we assume that a bank

[15]In the context of currency attack, Fujimoto (2014) also studies the symmetric threshold strategy where the attacking threshold of signals $x^*(\cdot)$ in a country i is a function of the realized signals of all other countries.

[16]As it will be clear from the analysis, this case is off the equilibrium path when the noises diminish.

[17]The reputational benefit may come from the fact that the creditor makes a "right decision". Rochet and Vives (2004) argue that the vast majority of wholesale deposits are held by collective investment funds, whose managers are compensated if they build a good reputation, and penalized otherwise. As we will show later, wholesale creditors receiving this small reputational payoff is also off the equilibrium path.

[18]Defining a bank run as a non-zero mass of withdrawals is an innocuous normalization. A run can be defined as the total withdrawals exceeding an alternative positive threshold. All results of our paper will qualitatively hold.

sells its assets if and only if it faces a bank run,[19] in which case, the bank sells its assets to the buyers who offer the highest price.

We assume that a large number of identical, deep-pocketed buyers participate in the market and that they are called into action only when a run happens. When no bank run occurs, the asset buyers will not have the opportunity to move, and the game between wholesale creditors and asset buyers ends. The buyers observe neither the aggregate state s nor any signals about the banks' cash flows. Thus, they cannot determine the exact quality of assets on sale. They can, nevertheless, observe the outcome of creditors' bank run game (i.e., the number of banks forced into asset sales) and can infer the quality of bank assets from the observation.[20]

An asset buyer will bid according to the creditors' optimal strategy as well as her observation of the number of bank runs. When called upon to move when M bank runs happen, the buyer forms rational beliefs about the aggregate state s and the quality of assets on sale. The asset buyer's strategy is a price schedule (P_1, P_2) with

$$M \longmapsto P_M, \ \ M = 1, 2. \tag{5}$$

We focus on symmetric strategies because the buyers are homogeneous and observe the same information. An equilibrium strategy (P_1^*, P_2^*) can be viewed as an inverse demand function for banks' assets. When bidding competitively to purchase banks' assets for price P_M^* in the contingency of M bank runs, the asset buyers break even in expectation.

2.4 Timing

The timing of the game is summarized in Figure 1. Events at $t = 1$ take place sequentially.

Figure 1: Timing of the game in a laissez-faire market

t = 0	t = 1	t = 2
Banks are established, with their portfolios and liability structures as given.	1. State s and (θ^1, θ^2) are realized sequentially. 2. Each creditor receives the noisy signals and decides whether to run on his own bank. 3. If any bank run occurs, buyers bid for and acquire assets on sale according to the number of runs observed.	1. Bank assets pay off. 2. Remaining obligations are settled.

[19] Diamond and Rajan (2011) provide an exposition why banks protected by the limited liability prefer not to sell their asset until runs happen, in which case the sale is too late and causes bank failures.

[20] For simplicity, we assume that asset buyers do not observe the precise size of runs in the case that only a fraction of creditors withdraw early. Also, such a partial run becomes a zero-probability event when the noises of creditors' signals approach zero.

3 Equilibrium bank runs and committed liquidity support

We solve the model using the concept of Perfect Bayesian Equilibrium.

Definition. *A PBE of the dynamic game consists of equilibrium strategy profile and a system of beliefs. (i) Creditors play a symmetric threshold strategy: a representative creditor j from a bank i withdraws if and only if his inside and private signal x^i_j about his own bank's fundamentals falls below an equilibrium threshold $x^*(y^{-i})$, with y^{-i} being the signal about the other bank's fundamentals that the creditor receives as an outsider. Asset buyers offer a price schedule (P^*_1, P^*_2) to purchase banks' assets on sale when observing M bank runs, $M \in \{1, 2\}$. (ii) Each creditor forms beliefs about the aggregate withdrawals in both banks based on his information (x^i_j, y^{-i}) and the equilibrium strategy profile described in (i). Each asset buyer forms beliefs about the qualities of banks' assets on sale and beliefs about the aggregate state based on her observation of M bank runs and the equilibrium strategy profile described in (i). (iii) The strategy profile described in (i) is sequentially rational, given the beliefs described in (ii).*

For an equilibrium strategy and fundamentals (θ^1, θ^2), an equilibrium outcome in a laissez-faire market can feature no bank runs and no asset liquidation (which we denote by *No Run*), or be summarized by a duplex (M, P^*_M), where $M \in \{1, 2\}$ is the number of bank runs and P^*_M is the prevailing market price for banks' assets.

3.1 Competitive bidding in the secondary asset market

In this section, we solve the asset buyers' bidding game. That is, given the creditors' strategy, what would be the secondary-market asset prices? Asset buyers observe neither fundamentals (θ^1, θ^2) nor the state s. Nevertheless, they form rational beliefs about the quality of assets on sale according to creditors' strategy and the number of bank runs observed. In a subgame of competitive bidding, buyers who believe creditors using a symmetric threshold strategy understand that a bank run happens if and only if the bank's cash flow is below a threshold fundamental θ^*. Asset buyers also Bayesian update their beliefs about the aggregate state s. Since the bank's cash flows are *i.i.d.* after the aggregate state is realized, more bank runs suggest that State B is more likely. Given their beliefs of creditors' equilibrium strategy and the observation of M bank runs, we can calculate buyers' posterior beliefs about the aggregate state s as follows (details in the Online Appendix):

$$\omega_M^B(\theta^*) \equiv Prob(s = B|\theta < \theta^*, M) = \frac{\left(\theta^* - \underline{\theta}_B\right)^M}{\left(\theta^* - \underline{\theta}_B\right)^M + \kappa\left(\theta^* - \underline{\theta}_G\right)^M}, \tag{6}$$

$$\omega_M^G(\theta^*) \equiv Prob(s = G|\theta < \theta^*, M) = \frac{\kappa\left(\theta^* - \underline{\theta}_G\right)^M}{\left(\theta^* - \underline{\theta}_B\right)^M + \kappa\left(\theta^* - \underline{\theta}_G\right)^M}, \tag{7}$$

where $\kappa \equiv \frac{\alpha}{1-\alpha}\left(\frac{\bar{\theta}-\underline{\theta}_B}{\bar{\theta}-\underline{\theta}_G}\right)^2$ is a constant and $M \in \{1, 2\}$. It should be noted that buyers' beliefs about s are endogenous to their beliefs about the creditors' equilibrium strategy.

When buyers bid competitively for banks' assets on sale, the equilibrium of the secondary market requires buyers' bids to be equal to the expected asset quality. Specifically, when M bank runs occur, the homogeneous buyers will offer the following competitive price:

$$P_M^* = \omega_M^B(\theta^*) \cdot E\left(\theta \middle| \theta < \theta^*, s = B\right) + \omega_M^G(\theta^*) \cdot E\left(\theta \middle| \theta < \theta^*, s = G\right).$$

For a given aggregate state s, the buyers perceive the average quality of the asset on sale to be

$$E\left(\theta \middle| \theta < \theta^*, s\right) = \int_{\underline{\theta}_s}^{\theta^*} \theta \cdot \frac{1}{\theta^* - \underline{\theta}_s} d\theta = \frac{\underline{\theta}_s + \theta^*}{2}, \quad s \in \{B, G\}.$$

Therefore, the competitive asset price can be written explicitly as follows:

$$P_M^* = \omega_M^B(\theta^*) \cdot \frac{\underline{\theta}_B + \theta^*}{2} + \omega_M^G(\theta^*) \cdot \frac{\underline{\theta}_G + \theta^*}{2} = \frac{E_s\left(\underline{\theta}_s|M\right) + \theta^*}{2}. \tag{8}$$

Expression $E_s(\underline{\theta}_s|M) = \omega_M^B(\theta^*) \cdot \underline{\theta}_B + \omega_M^G(\theta^*) \cdot \underline{\theta}_G$ represents the expected lower bound of θ, based on the observation of M runs and the belief that a bank fails if and only if its cash flow is lower than a critical θ^*.[21]

Creditors' strategy affects the secondary-market price in two ways. First, a higher value of critical signal $x^*(y)$ and the associated θ^* directly determine the average quality of the assets on sale. Second, θ^* affects buyers' perception of the aggregate state. For a given number of runs, a more pessimistic strategy on the creditors' side (i.e., a higher $x^*(y)$ and the resulting higher

[21]Our model does not feature asset fire sales. Since the buyers pay the expected payoff of the asset given their information set, no welfare loss emerges due to the change of ownership of the asset. This differs from classic views of asset fire sales, such as in Shleifer and Vishny (1992). When the seller of the asset can make better use of the capital as compared to the potential buyers, the asset sale itself generates a negative impact on social welfare.

θ^*) is associated with a more optimistic perception of the aggregate state s (i.e., a higher ω_M^G). Both channels imply that higher $x^*(y)$ and θ^* are associated with higher asset prices.

Also, any break-even price P offered by asset buyers must belong to $[\underline{P}, qD_2)$, with $\underline{P} \equiv (\underline{\theta}_B + D_2)/2$. Intuitively, if the price is greater than qD_2, early liquidation will not hurt a bank's solvency so that its creditors would not run in the first place.[22] On the other hand, since at least all fundamentally insolvent banks (i.e., those with $\theta < D_2$) will be liquidated, the worst possible average asset quality is $(\underline{\theta}_B + D_2)/2$. Parametric assumption (3) guarantees $qD_2 > \underline{P}$, so that the set $[\underline{P}, qD_2)$ is non-empty. We summarize the result in Lemma 1.

Lemma 1. *When asset buyers believe that creditors follow a symmetric threshold strategy and that a bank fails if and only if its cash flow is lower than a threshold, their break-even price P cannot be greater than or equal to qD_2, nor can it be smaller than $\underline{P}$.*

Proof. See Appendix B.1. □

Since the equilibrium asset price P_M^* characterized by the equation (8) makes asset buyers break even in expectation in each contingency of $M \in \{1, 2\}$ bank runs, it follows immediately that $P_M^* \in [\underline{P}, qD_2)$. This restriction on the range of equilibrium asset prices will facilitate the solution of the bank run game in the next section.

Furthermore, it holds that $P_M^* \geq \underline{P} > D_1$,[23] so that a bank can always repay its $t = 1$ liabilities and does not fail on the intermediate date. A run, however, does accelerate the bank failure because liquidation losses lead to bankruptcy at $t = 2$. Specifically, while a partial liquidation can generate sufficient cash to pay early withdrawals and the bank does not immediately fail at $t = 1$, the cash flow from the residual portfolio will be insufficient to cover the remaining liabilities at $t = 2$, making a bank that is otherwise solvent fail at $t = 2$.[24]

3.2 The bank run game

We solve the creditors' bank run game by examining the strategy of a representative creditor j from a bank i, $i = 1, 2$. We derive the creditor's threshold strategy as a best response to other players' equilibrium strategies $x^*(\cdot)$ and (P_1^*, P_2^*).

[22]For an asset price equal to qD_2, one can show that any run will reduce a bank's asset and liabilities by the same amount, resulting in a neutral impact of runs on the solvency of the bank.

[23]Note that for $q < 1$, inequality (2) implies $D_2 > 2D_1$, because $D_2 = D_1/q + F > D_1 + F > 2D_1$.

[24]Similar to Morris and Shin (2016), even if a bank survives $t = 1$ runs, it would be doomed to fail at $t = 2$. The funding liquidity risk is captured by a higher ex-ante probability of bank failure and the fact the survival threshold is higher than the solvency threshold. This feature of no interim date failure also emerges in Ahnert et al. (2019).

Facing a given asset price $P \in [\underline{P}, qD_2)$, the representative creditor will withdraw if and only if the aggregate withdrawal in his bank, L^i, exceeds a critical level. To see this, note that when an L^i fraction of its creditors withdraw, the bank i faces a liquidity demand of $L^i D_1$ and needs to liquidate a $\lambda^i = L^i D_1 / P$ fraction of its assets, where $\lambda^i \in (0, 1)$ because $P > D_1$. After the partial liquidation, the bank will fail at $t = 2$ if and only if its remaining assets fall below its remaining liabilities, i.e., $(1 - \lambda^i)\theta^i < F + (1 - L^i)(1 - E - F)r_D$. In other words, the bank fails at $t = 2$ if and only if L^i exceeds a critical value L^c:

$$L^i > \frac{P \cdot (\theta^i - D_2)}{D_1 \cdot (\theta^i - P/q)} \equiv L^c(\theta^i, P). \tag{9}$$

If the representative creditor withdraws, his payoff will be $W_{run} = D_1$ since the bank does not fail at $t = 1$. If he waits instead, his payoff W_{wait} depends on L^i, θ^i and P. He would receive D_1/q if the bank survives at $t = 2$, and 0 otherwise. Denote $DW(L^i, \theta^i, P) \equiv W_{wait} - W_{run}$ as the creditor's payoff difference from the two actions, we have

$$DW(L^i, \theta^i, P) = \begin{cases} (1-q)D_1/q & L^i \in [0, L^c(\theta^i, P)] \\ -D_1 & L^i \in (L^c(\theta^i, P), 1]. \end{cases} \tag{10}$$

The creditor strictly prefers 'wait' ('withdraw') if L^i is lower (higher) than $L^c(\theta^i, P)$. Given the strong strategic complementarity, a bank run game with complete information can feature multiple equilibria.

With incomplete information, the creditor cannot directly observe his bank's cash flow θ^i or the aggregate withdrawal L^i. The asset price will also be endogenous to the number of runs. To determine his optimal action, the creditor has to form beliefs about θ^i, L^i, and P, based on his information (x^i_j, y^{-i}) and the other players' equilibrium strategies.

In the remaining of this section, we take a step-by-step approach to derive the equilibrium of the incomplete information game: we first analyze a case without aggregate uncertainty for an illustrative purpose in Section 3.2.1, and analyze the fully-fledged model with the aggregate uncertainty in Section 3.2.2.

3.2.1 Equilibrium without the aggregate uncertainty

Suppose that there is no aggregate uncertainty and $\underline{\theta}_G = \underline{\theta}_B = \underline{\theta}$. We solve the game backward, starting with the asset market. With no aggregate uncertainty and banks' cash flows independently distributed, creditors' run in one bank provides no information about the other bank's fundamentals. The asset buyers thus offer a single price P independent of the number of runs observed. In other words, their strategy features a price schedule $(P_1, P_2) = (P, P)$, suggesting the demand for banks' assets to be perfectly elastic. As discussed in Section 3.1, a candidate equilibrium price P^* must satisfy the zero-profit condition (8). Without aggregate uncertainty, $E_s(\underline{\theta}_s|M)$ degenerates to $\underline{\theta}$, and the condition becomes

$$P^* = \frac{\underline{\theta} + \theta^*}{2}. \tag{11}$$

To solve the bank run game, we first show the existence of lower and upper dominance regions. When the representative creditor observes $x_j^i < x^L \equiv D_2 - \epsilon$ and knows his bank's fundamentals below $\theta^L \equiv D_2$, it is a dominant strategy for him to withdraw early, independent of his beliefs about L^i and for any asset price $P \in [\underline{P}, qD_2)$. Similarly, when the creditor observes $x_j^i > x^U \equiv F/(1 - D_1/\underline{P}) + \epsilon$ and learns his bank's fundamentals above $\theta^U \equiv F/(1 - D_1/\underline{P})$, it is a dominant strategy for the creditor to wait, independent of his beliefs about L^i and for any asset price $P \in [\underline{P}, qD_2)$ (details in Appendix A.1).[25]

For an intermediate private signal $x_j^i \in [x^L, x^U]$, the representative creditor's optimal action depends on the asset price P and his beliefs about the aggregate withdrawal in his bank, L^i. When all other creditors in the bank i observe y^{-i} and take $x^*(y^{-i})$ as the critical signal, one can derive $L^i(\theta^i, x^*(y^{-i}))$ as the aggregate withdrawal faced by the bank i with a cash flow θ^i (details in Appendix A.2):

$$L^i(\theta^i, x^*(y^{-i})) = \begin{cases} 1 & \theta^i < x^*(y^{-i}) - \epsilon \\ \frac{x^*(y^{-i}) - \theta^i + \epsilon}{2\epsilon} & x^*(y^{-i}) - \epsilon \le \theta^i \le x^*(y^{-i}) + \epsilon \\ 0 & \theta^i > x^*(y^{-i}) + \epsilon. \end{cases}$$

Based on his signals, the representative creditor forms a posterior belief $\theta^i|_{x_j^i} \sim U(x_j^i - \epsilon, x_j^i + \epsilon)$ about his bank's fundamentals. Consequently, he expects the following aggregate withdrawal in his own bank:

$$L^i(x_j^i, y^{-i}) = E\left[L^i(\theta^i, x^*(y^{-i}))\middle| x_j^i, y^{-i}\right] = \int_{x_j^i - \epsilon}^{x_j^i + \epsilon} L^i(\theta^i, x^*(y^{-i})) \cdot \frac{1}{2\epsilon} \cdot d\theta^i. \tag{12}$$

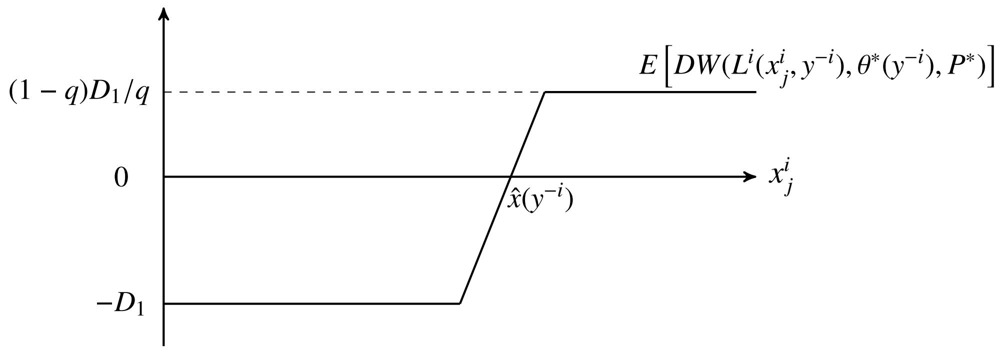
[25] The upper dominance region is non-empty provided $\overline{\theta} > F/(1 - D_1/\underline{P})$.

The creditor anticipates his bank to sell its assets at the price P^* if runs happen. Therefore, by condition (9), the creditor expects his bank with a critical fundamental $\theta^*(y^{-i})$ to fail if and only if

$$L^i > L^c(\theta^*(y^{-i}), P^*) = \frac{P^* \cdot (\theta^*(y^{-i}) - D_2)}{D_1 \cdot (\theta^*(y^{-i}) - P^*/q)}. \tag{13}$$

We now calculate the representative creditor's expected payoff difference conditional on his signals (x_j^i, y^{-i}) as $E\left[DW(L^i, \theta^i, P) \middle| x_j^i, y^{-i}\right]$, which can be further expressed with expressions (12) and (13) as $E\left[DW(L^i(x_j^i, y^{-i}), \theta^*(y^{-i}), P^*)\right]$. For a given y^{-i}, we illustrate in Figure 2 the expected payoff difference as a function of x_j^i. The representative creditor's best response to the other creditors' threshold strategy $x^*(y^{-i})$ is also a threshold strategy: to withdraw if and only if $x_j^i < \hat{x}(y^{-i}) = x^*(y^{-i}) - 2\epsilon \cdot [L^c(\theta^*(y^{-i}), P^*) - q]$ (details in Appendix B.2).

Figure 2: Payoff differences and the decision to withdraw



A symmetric threshold equilibrium requires $\hat{x}(y^{-i}) = x^*(y^{-i})$. So the equilibrium critical cash flow $\theta^*(y^{-i})$ must satisfy $L^c(\theta^*(y^{-i}), P^*) = q$, which implies

$$\theta^*(y^{-i}) = \frac{D_2 - D_1}{1 - qD_2/P^*}. \tag{14}$$

Furthermore, the bank i with fundamentals $\theta^i = \theta^*(y^{-i})$ is on the verge of bankruptcy, so that the aggregate withdrawal $L^i(\theta^*(y^{-i}), x^*(y^{-i})) = (x^*(y^{-i}) - \theta^*(y^{-i}) + \epsilon)/2\epsilon$ must equal $L^c(\theta^*(y^{-i}), P^*)$, which further equals q in the equilibrium. This defines the equilibrium threshold signal:

$$x^*(y^{-i}) = \theta^*(y^{-i}) + (2q - 1)\epsilon. \tag{15}$$

In a Perfect Bayesian equilibrium, the asset buyers' belief about banks' critical cash flow must be consistent with the one associated with creditors' equilibrium strategy. The following condition must hold:

$$\theta^*(y^{-i}) = \theta^*. \tag{16}$$

The expressions (15) and (16) imply that the equilibrium critical cash flow $\theta^*(y^{-i})$ and the equilibrium critical signal $x^*(y^{-i})$, if exist, are constant and do not depend on y^{-i}. Proposition 1 establishes the existence and the uniqueness of the equilibrium.

Proposition 1. *Without aggregate uncertainty, the game has a unique equilibrium: a wholesale creditor of a bank i withdraws if and only if his private signal falls below x^*, which is a unique threshold independent of y^{-i}. The asset buyers offer a price P^* to buy banks' assets, independent of the number of runs.*

Proof. See Appendix B.2 □

Our model predicts two-way feedback between bank runs and distressed asset prices even in the absence of the aggregate uncertainty. The equilibrium is unique and stable despite the two-way feedback. Intuitively, if creditors take a more optimistic strategy than the equilibrium one, they rationally anticipate the asset buyers to bid a lower price P^* according to (11). The lower P^*, however, implies an aggravated coordination problem by (14), which, in turn, restores the equilibrium threshold strategy.[26]

A few comments are in order. First, our model differs from classic global-game-based bank run models such as Rochet and Vives (2004) and Vives (2014) because the liquidation value of banks' assets is endogenous: creditors in our model are forward-looking and understand the impact of their decisions to run on asset prices. The two-way feedback is related to Liu (2016) who studies the interaction between bank runs and rising interbank market rates when there is a limited supply of cash. We show that given the lack of information, bank runs depress asset prices even if the supply of cash is perfectly elastic.

Second, traditional LoLR policies that target only solvent-but-illiquid banks are infeasible when a central bank does not possess accurate information on banks' solvency. An informationally constrained central bank cannot offer a better price than what private market participants are willing to pay — at least not without incurring expected losses. The presence of aggregate uncertainty would open the possibilities for alternative policy interventions that promote financial stability.

[26] It is worth noticing that the information asymmetry in our model does not generate standard adverse selection problems where lower prices are associated with lower average qualities. Since banks in our model are forced into asset sales rather than strategically choose to do so, a lower asset price is associated with a *higher* average quality.

3.2.2 Equilibrium with the aggregate uncertainty

We now characterize the equilibrium of the fully-fledged model with both idiosyncratic and aggregate risks. Different from the case without aggregate uncertainties, the equilibrium asset price P_M^* now depends on the number of runs M, which conveys information about the aggregate state s. In deciding whether to run, a creditor needs to anticipate the would-be asset price if his bank is forced into liquidation. To do that, the creditor needs to form rational expectations about the total withdrawals, not only in his own bank but also in the other bank.

For a sufficiently low or high y^{-i}, the representative creditor can tell whether the creditors in the other bank receive private signals that fall into the dominance regions. For example, when $y^{-i} < y^L \equiv x^L - \eta - \epsilon$, the creditor knows that all creditors in the bank $-i$ receive private signals lower than x^L and will withdraw. So the creditor expects $L^{-i}(x_j^i, y^{-i}) = 1$ independent of his private signal x_j^i. Similarly, for $y^{-i} > y^U \equiv x^U + \eta + \epsilon$, the representative creditor knows that all creditors in the bank $-i$ receive private signals in the upper dominance region, and expects $L^{-i}(x_j^i, y^{-i}) = 0$ independent of x_j^i. We summarize the result in Lemma 2.

Lemma 2. *When observing $y^{-i} < y^L \equiv x^L - \eta - \epsilon$, the representative creditor expects $L^{-i}(x_j^i, y^{-i}) = 1$, independent of his private signal x_j^i. When observing $y^{-i} > y^U \equiv x^U + \eta + \epsilon$, the creditor holds a belief that $L^{-i}(x_j^i, y^{-i}) = 0$, independent of his private signal x_j^i.*

When observing $y^{-i} > y^U$, the representative creditor anticipates the asset price to be P_1^* if his own bank is forced into liquidation because a run will not happen to the other bank. Similarly, upon observing $y^{-i} < y^L$, the representative creditor anticipates a run on the other bank, so that his own bank will have to sell assets for the price P_2^* if a run happens. With such expectations of asset prices, we can derive the representative creditor's best response following the same procedure as in Section 3.2.1, except that the asset price is now given by the equation (8) with $M = 1$ for $y^{-i} > y^U$, and with $M = 2$ for $y^{-i} < y^L$. The asset price, the bank i's creditors' threshold signal, and the bank's critical cash flow, solve the following system of equations:

$$\begin{cases} P_M^* = \omega_M^B(\theta_M^*) \cdot \frac{\theta_B + \theta_M^*}{2} + \omega_M^G(\theta_M^*) \cdot \frac{\theta_G + \theta_M^*}{2} \\ x^*(y^{-i}) = \theta^*(y^{-i}) + (2q-1)\epsilon \\ \theta^*(y^{-i}) = \frac{D_2 - D_1}{1 - qD_1/P_M^*} \\ \theta^*(y^{-i}) = \theta_M^* \end{cases} \quad \text{with} \quad M = \begin{cases} 1 & \text{if } y^{-i} > y^U \\ 2 & \text{if } y^{-i} < y^L. \end{cases} \tag{17}$$

We establish in Lemma 3 the existence and uniqueness of P_M^*, x_M^*, and θ_M^*, $M \in \{1,2\}$, and show that $P_1^* > P_2^*$, $x_1^* < x_2^*$, and $\theta_1^* < \theta_2^*$. Intuitively, asset buyers form more pessimistic beliefs about the aggregate state s when observing more bank runs. As a result, they will offer a lower price, which, in turn, gives the creditors greater incentives to withdraw and pushes up the threshold cash flow for banks to survive runs.

Lemma 3. *When observing $y^{-i} > y^U$, a creditor in a bank i withdraws if and only if his private signal falls below x_1^* and expects his bank to liquidate its assets for a price P_1^*. The bank fails if and only if its cash flow is below θ_1^*, with P_1^*, x_1^*, and θ_1^*, being the unique solution to the system of equations (17) for $M = 1$. Similar, P_2^*, x_2^*, and θ_2^*, jointly solve the system of equations for $y^{-i} < y^L$ and $M = 2$. It holds that $P_1^* > P_2^*$, $x_1^* < x_2^*$, and $\theta_1^* < \theta_2^*$.*

Proof. See Appendix B.3. □

Lemma 3 implies that the creditors' equilibrium threshold signal $x^*(y^{-i})$ must be a step function with only two possible values. Since the asset buyers can only observe the number of bank runs, $M \in \{1,2\}$, and offer a corresponding price P_M^*, the creditors will choose their threshold strategy in expectation of one of the two equilibrium prices. Since we have established x_1^* and x_2^* as the two values of $x^*(y^{-i})$, for $y^{-i} > y^U$ and $y^{-i} < y^L$ respectively, the non-increasing function $x^*(y^{-i})$ must be bounded between x_1^* and x_2^*. Furthermore, there must exist a point of discontinuity $\hat{y} \in [y^L, y^U]$ such that $x^*(y^{-i}) = x_1^*$ for $y^{-i} \geq \hat{y}$, and $x^*(y^{-i}) = x_2^*$ for $y^{-i} < \hat{y}$. As y^{-i} rises above the cutoff value $\hat{y}$, the creditor's expected asset price changes from P_2^* to P_1^*.

To fully characterize creditors' equilibrium strategy, we now derive $x^*(y^{-i})$ for $y^{-i} \in [y^L, y^U]$, in which case creditors in the bank $-i$ are no longer seen to have a dominant action. We establish in Proposition 2 that a unique equilibrium exists for the game under aggregate uncertainty and illustrate the creditors' equilibrium threshold signal in Figure 3.

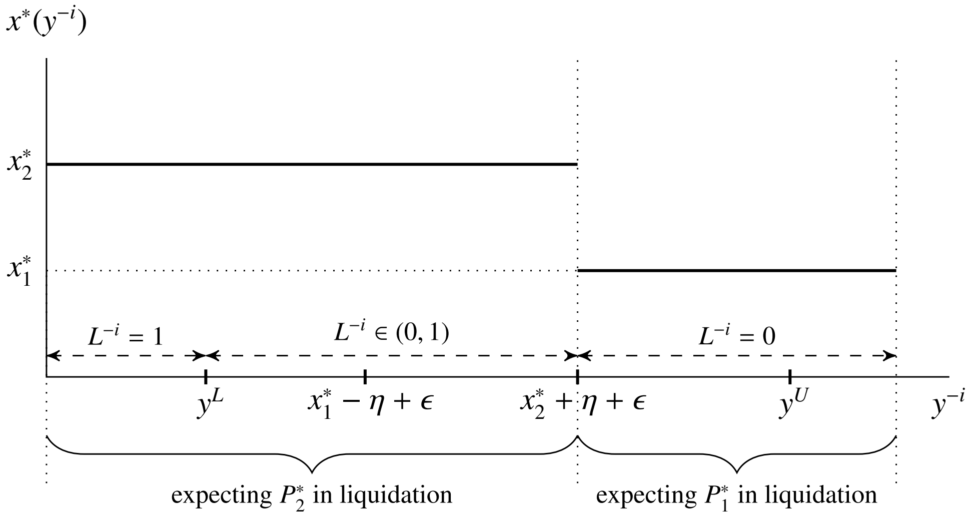
Proposition 2. *With aggregate uncertainty, the game has a unique equilibrium: a wholesale creditor of a bank i withdraws if and only if his private signal falls below $x^*(y^{-i})$, with*

$$x^*(y^{-i}) = \begin{cases} x_2^* & y^{-i} < x_2^* + \eta + \epsilon \\ x_1^* & y^{-i} \geq x_2^* + \eta + \epsilon. \end{cases} \tag{18}$$

The asset buyers offer price P_M^* when observing M bank runs, $M = 1, 2$.

Proof. See Appendix B.4. □

Figure 3: The equilibrium threshold strategy



To establish Proposition 2, we first derive the representative creditor's expectation of L^{-i} for those relatively high and low values of y^{-i} in the intermediate range. In particular, the creditor expects $L^{-i}(x_j^i, y^{-i}) = 0$ when observing $y^{-i} > x_2^* + \eta + \epsilon$, independent of his private signal. Indeed, such an observation reveals that all creditors in the bank $-i$ observe private signals no less than x_2^*. Therefore, all those creditors will wait even if they follow the most aggressive strategy with the threshold signal x_2^*. The representative creditor then expects $L^{-i} = 0$ and sets his threshold signal to $x^*(y^{-i}) = x_1^*$. Similarly, when observing $y^{-i} < x_1^* - \eta + \epsilon$, the creditor knows that $\theta^{-i} < x_1^* + \epsilon$ must be true, and that a positive mass of creditors in the bank $-i$ will withdraw even if they follow the least aggressive strategy with a threshold signal x_1^*. As a result, the representative creditor expects $L^{-i} > 0$ and sets his threshold signal to x_2^*. Therefore, we know the point of discontinuity must be in the range of $[x_1^* - \eta + \epsilon, x_2^* + \eta + \epsilon]$.

We now sketch the proof for why the point of discontinuity must be $x_2^*+\eta+\epsilon$, and provide the full proof by contradiction in Appendix B.4. Suppose that there is an alternative discontinuity point $\hat{y} < x_2^* + \eta + \epsilon$. When observing a $y^{-i} \in (\hat{y}, x_2^* + \eta + \epsilon)$, the bank i's creditors would follow a threshold signal x_1^*, which must be rationalized by the expectation that their bank — when forced into liquidation — will sell its assets for the price P_1^*. In other words, the creditors must expect a run not to occur to the bank $-i$. We can also make the following two observations. First, when a representative creditor from the bank i observes $x_j^i = x_1^*$ and expects a run on his own bank, the creditor cannot exclude the possibility that the bank $-i$'s creditors run according to the threshold x_2^*. This is because the representative creditor knows that $\theta^i < x_1^* + \epsilon$ as long as he does not receive the lowest private signal in his bank. Therefore, he cannot exclude the possibility that the bank $-i$'s creditors observe an outside signal $y^i < x_1^* - \eta + \epsilon$, which implies those creditors following the threshold signal x_2^*. Second, since the representative creditor also receives the outside signal $y^{-i} < x_2^* + \eta + \epsilon$, he perceives with a positive probability that $\theta^{-i} < x_2^* + \epsilon$, in which case those lowest private signals in the bank $-i$ would be lower than x_2^*. Combining these two observations, the representative creditor cannot exclude the possibility that a run also happens to the bank $-i$. Such a possibility, however, contradicts the expectation of the price P_1^*.

Our fully-fledged model with two groups of creditors and two-dimensional signals features a unique equilibrium despite the aggregate uncertainty. To gain some intuition for the uniqueness, note that the signal y^{-i} received by a bank i's creditors is about the other bank's fundamentals, which differs from the classic models where both private and public signals are about the same bank's cash flow.[27] Therefore, on top of the coordination game within each bank, the two groups of creditors in our model also play a cross-bank coordination game on the number of runs. Creditors in a bank i need to rely on both their private signals (to Bayesian update the threshold signal $x^*(y^i)$ adopted by the bank $-i$'s creditors) and the outside signal (to Bayesian update the range of private signals received by the bank $-i$'s creditors) to form rational beliefs about whether a run would happen to the bank $-i$. Since each creditor has its own posterior about $x^*(y^i)$, there is a lack of common knowledge on whether a run will happen to the bank $-i$, which results in the uniqueness of the equilibrium.

A couple of comments are due. First, the unique equilibrium of our model features strategic complementarities between creditors from the two banks. Observing a low outside signal y^{-i} about the bank $-i$'s cash flow, creditors in the bank i know that the bank $-i$'s creditors are likely to run. As a consequence, the bank i's creditors expect their bank to face the price P_2^* if it is forced into liquidation, which increases their incentives to withdraw.[28]

Second, if a regulator intervenes by purchasing the banks' assets after observing that bank runs have happened, the regulator cannot offer prices higher than the private asset buyers — at

[27]In models such as Morris and Shin (2001) and Hellwig (2002), the global game refinement could fail to predict a unique equilibrium as the public signal becomes increasingly precise as compared to the private signals. As a result, the public signal brings players' perfect coordination back to the game. In contrast, the uniqueness of the equilibrium of our model does not rely on the relative magnitude between ϵ and η.

[28]Also in a global-games framework, Ahnert and Bertsch (2020) model how a bank failure serves as a 'wake-up call' and triggers information acquisition about other banks' exposure to the aggregate risk. In contrast, we emphasize how the information constraint faced by asset buyers leads to liquidation losses, and how such losses trigger runs from forward-looking creditors who understand the price impact of bank asset liquidation.

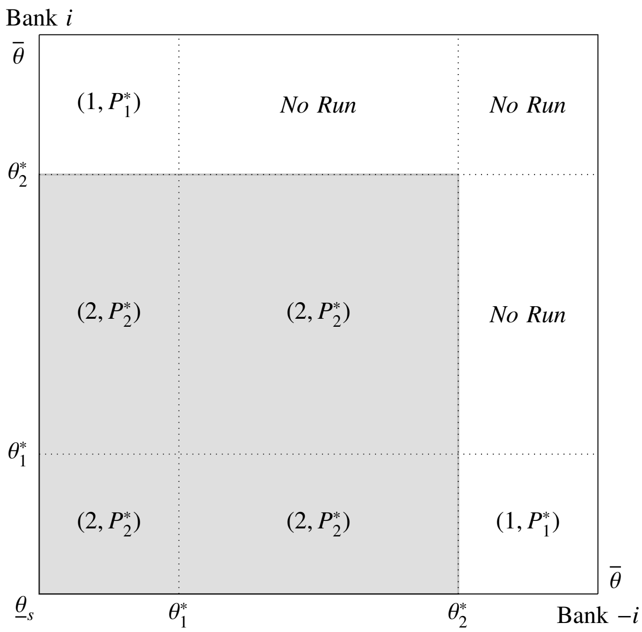
least not without incurring expected losses. This is because other than the coordination failure, the only friction in our model is the lack of information on asset qualities. If the regulator only moves after observing the number of runs M, the efficiency of the policy intervention will be bounded by the market allocation.

Equilibrium outcomes under the aggregate uncertainty

We now give a complete characterization of how the equilibrium outcome depends on the banks' fundamentals. Since the relative magnitude of η and ϵ does not affect any main results, we focus on a limiting case where $\epsilon \to 0$ and $\eta \to 0$.[29] As the uncertainties about both banks' fundamentals diminish, we have the critical cash flows $\theta_1^* = lim_{\epsilon\to 0} x_1^*$ and $\theta_2^* = lim_{\epsilon\to 0} x_2^*$, which make a unique partition of the set of bank fundamentals as shown in Figure 4.

Assumption. *In the remaining of the paper, we focus on the case where $\epsilon \to 0$ and $\eta \to 0$.*

Figure 4: Bank fundamentals and the equilibrium outcomes



The equilibrium of our model suggests a potential for financial contagion. When the bank $-i$'s fundamentals are weak, the bank i's creditors would observe a low signal y^{-i} and act more aggressively, pushing up the critical fundamental for their own bank's survival. The bank i's exposure to the risk of contagion can be quantified as the increase in its probability of failure

[29]For its tractability, it is common to study the limiting case in the literature: e.g., see Liu and Mello (2011).

when the bank $-i$'s fundamentals weaken:

$$Prob(Bank\ \ i\ \ fails \mid \theta^{-i} < \theta_2^*) - Prob(Bank\ \ i\ \ fails \mid \theta^{-i} \geq \theta_2^*) = Prob(\theta_1^* < \theta^i < \theta_2^*). \tag{19}$$

When both banks' fundamentals fall below θ_2^*, the equilibrium strategy suggests runs occurring to both banks. We indicate the equilibrium outcomes in Figure 4 and depict the regions of fundamentals where both banks fail in grey, a scenario that we dub as '*systemic bank runs*'.

Corollary 1. *Systemic bank runs happen when both banks' fundamentals are below* θ_2^*.

Corollary 1 shows that a systemic crisis can emerge in a laissez-faire market — even if the fundamentals are strong, e.g., both banks' fundamentals only marginally below θ_2^*. From an ex-ante perspective, the probability of systemic bank runs can be computed as follows:

$$\text{SYS}(\theta_2^*) \equiv Prob(\theta^i \leq \theta_2^*, \theta^{-i} \leq \theta_2^*) = \alpha \cdot \left(\frac{\theta_2^* - \underline{\theta}_G}{\bar{\theta} - \underline{\theta}_G}\right)^2 + (1-\alpha) \cdot \left(\frac{\theta_2^* - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right)^2.$$

Since a systemic crisis is particularly detrimental for its costly resolution and negative impacts on the broader economy,[30] we consider liquidity interventions that would reduce the range of fundamentals where systemic runs can happen.

3.3 Policy Intervention

In this section, we analyze a regulator's options for policy interventions when she strives to simultaneously achieve the following three desirable goals: (1) to mitigate systemic risks, (2) to make no loss in the intervention, and (3) to save no insolvent bank. The regulator tries to avoid making losses, as such losses would constitute a transfer to banks' claim holders and make the policy politically unpopular. It is also understood that the regulator should also avoid using liquidity interventions to tackle insolvency problems, since assisting insolvent banks would create zombie lending and weaken the ex-ante discipline.

3.3.1 Intervention without information constraints: a benchmark

To provide a benchmark for the policy analysis, we start with a case where the regulator holds information on banks' solvency. With such information, the regulator can offer to pur-

[30]A systemic banking crisis can threaten essential payment services and causes system-wide disintermediation, e.g., a credit crunch and the loss of soft information on borrowers. See Laeven et al. (2010) for the real and fiscal cost of systemic banking crises. An individual bank failure, on the other hand, can be much less concerning.

chase assets *only from solvent banks* for a price $P \geq qD_2$. Knowing the offer, no creditors of a solvent bank will withdraw early. To see this, note that even if all wholesale creditors withdraw at $t = 1$, the bank only needs to sell a $\lambda = D_1/P$ fraction of its assets and can stay solvent at $t = 2$, because $(1 - \lambda)\theta \geq [1 - D_1/(qD_2)] \cdot D_2 = F$. Consequently, no creditors would run in the first place. The regulator eliminates the risk of runs without actually purchasing assets, and the targeted policy also saves no insolvent bank.

Proposition 3. *With information on banks' solvency, the regulator can offer to purchase assets only from solvent banks for a price $P \geq qD_2$, which will costlessly reduce the critical cash flow for a bank's survival to D_2. In other words, only insolvent banks will fail.*

The policy described in Proposition 3 simultaneously achieves all three desirable goals outlined at the beginning of this section, and, therefore, provides a benchmark for our policy analysis. In this benchmark, no coordination failure occurs among a solvent bank's creditors. Intuitively, since a solvent bank's assets are no longer pooled with those of insolvent banks', the increase in the price of the former will remove the first-mover advantage for creditors who run on such a bank. Consequently, runs are confined to the lower dominance region and occur only to the fundamentally insolvent banks. This benchmark result echos how deposit insurance works in Diamond and Dybvig (1983): once a bank is known free of insolvency risks, liquidity support can costlessly eliminate bank runs.

3.3.2 Intervention with information constraints: the committed liquidity support

Once the regulator is informationally constrained, the ideal allocation $\theta^* = D_2$ can no longer be achieved. We consider an ex-ante liquidity intervention with mutual commitments from the regulator and banks.[31] Formally, we assume that the regulator intervenes by committing to purchasing banks' assets for a price $P_A \geq P_2^*$ in case any bank is forced into liquidation at $t = 1$, and the banks commit to raising liquidity by selling their assets to the regulator when experiencing runs. The support price P_A is *pre-emptive*, in the sense that it is set up before the realization of the systematic and idiosyncratic risks, and therefore before the observation of any actual bank run. In line with the three policy goals, we assume that the regulator solves the

[31] As we will show in Section 4.2, a liquidity arrangement with the regulator's unilateral commitment cannot achieve all three goals simultaneously, and in particular, requires the regulator to bear losses from the intervention.

following program:

$$\begin{aligned}
\min_{P_A \geq P_2^*} \quad & \text{SYS}(P_A) \\
\text{s.t.} \quad & V_A(P_A) \geq 0 & (20)\\
& \theta^*(P_A) \geq D_2. & (21)
\end{aligned}$$

The regulator's objective is to minimize the probability of systemic bank runs $\text{SYS}(P_A)$ by setting the support price P_A. With $V_A(P_A)$ denoting the regulator's expected payoff from purchasing bank assets at a price P_A, the constraint (20) requires the regulator to make no loss or net transfer to banks. With $\theta^*(P_A)$ denoting the equilibrium threshold of cash flow under the regulator's intervention, the constraint (21) states that the liquidity support should not save any insolvent bank.[32]

Observing a committed price $P_A \geq P_2^*$, creditors no longer need to consider the price impact of bank runs. The bank run game can be solved as in Rochet and Vives (2004). In the limiting case where ϵ and η approach zero, the creditors' equilibrium threshold signal and the banks' critical fundamentals become

$$\theta^*(P_A) = x^*(P_A) = \frac{D_2 - D_1}{1 - qD_1/P_A}. \tag{22}$$

The risk of systemic bank failures under the intervention can be expressed as follows:

$$\text{SYS}(P_A) = \alpha \cdot \left(\frac{\theta^*(P_A) - \underline{\theta}_G}{\bar{\theta} - \underline{\theta}_G}\right)^2 + (1-\alpha) \cdot \left(\frac{\theta^*(P_A) - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right)^2. \tag{23}$$

When choosing a P_A at $t = 0$, the regulator needs to calculate her expected payoff, by forming rational expectations about the possible number of runs and the associated probabilities of $M = 1, 2$. The regulator's expected payoff can be expressed as follows.

$$V_A(P_A) = \sum_{s=G,B} Pr(s) \cdot \left(\sum_{M=1}^{2} Pr(\theta < \theta^*(P_A)|s)^M \cdot Pr(\theta > \theta^*(P_A)|s)^{2-M} \cdot C_2^M \cdot M \cdot \frac{D_1}{P_A} \cdot \pi(P_A|s)\right). \tag{24}$$

Here $\pi(P_A|s) = [\underline{\theta}_s + \theta^*(P_A)]/2 - P_A$ is the regulator's expected payoff from purchasing one unit of assets from a bank for the price P_A in a given aggregate state s. In the equilibrium,

[32] Our modeling of the regulator's objective function is also in line with the theory literature, e.g., Vives (2014) and Morris and Shin (2016). Both papers distinguish between a bank's liquidity and insolvency risks and assume the regulator to focus on containing the risks. In our paper, the liquidity intervention aims to reduce the risk of systemic crises by preventing contagious bank runs while making no attempt to limit insolvency.

a bank with a cash flow $\theta \in [\underline{\theta}_s, \theta^*(P_A))$ will experience creditors' run and will have to sell D_1/P_A proportion of its portfolio to meet D_1 amount of withdrawals. Therefore, $M \cdot (D_1/P_A) \cdot \pi(P_A|s)$ denotes the regulator's total payoff of purchasing assets from M banks, $M \in \{1, 2\}$. Since $Pr(\theta < \theta^*(P_A)|s)^M \cdot Pr(\theta > \theta^*(P_A)|s)^{2-M}$ is the probability that M and only M banks are forced into liquidation in a given aggregate state s, the expression in the parentheses denotes the regulator's expected payoff in the state s. We characterize the solution of the regulator's program in Proposition 4.

Proposition 4. *There exists a unique $P_A^* \in (P_2^*, P_1^*)$ such that the regulator can break even with the intervention, i.e., $V_A(P_A^*) = 0$. With both $SYS(P_A)$ and $V_A(P_A)$ monotonically decreasing in P_A, the regulator optimally commits to purchasing banks' assets for the break-even P_A^* and reduces the risk of systemic bank runs from $SYS(\theta_2^*)$ to $SYS(P_A^*)$.*

Proof. See Appendix B.5. □

Offering the price $P_A^* \in (P_2^*, P_1^*)$ at $t = 0$ allows the regulator to *break even across possible posterior beliefs of the state s*, whereas the private asset buyers who bid ex-post at $t = 1$ have to *break even within a given belief of the state s*. To highlight this difference, we define $\Pi_M(P)$ as the expected payoff from purchasing one unit of bank assets for a *given* price $P \in (P_2^*, qD_2)$ in the contingency of M bank runs. Denoting the corresponding critical cash flow by $\theta^*(P)$, we have

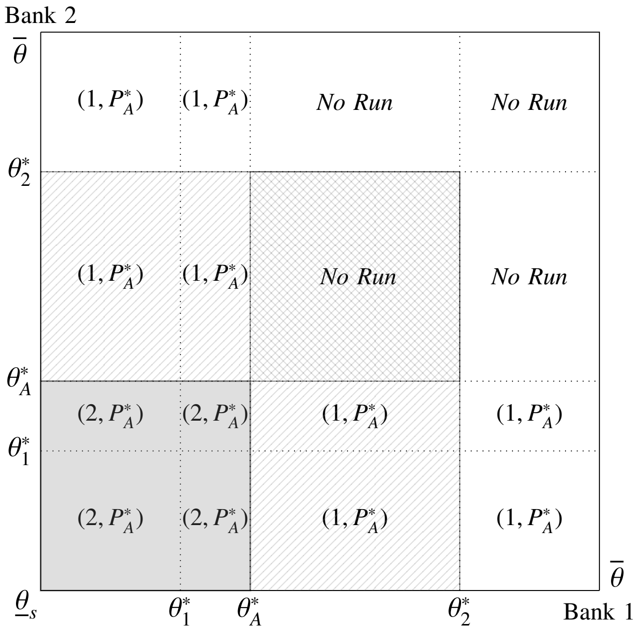
$$\Pi_M(P) = \omega_M^B(\theta^*(P)) \cdot \pi(P|s = B) + \omega_M^G(\theta^*(P)) \cdot \pi(P|s = G).$$

Here, $\omega_M^s(\theta^*(P))$ is the posterior belief about the state s upon the observation of M bank runs when the bank's assets are sold for a price P. One can verify that $\Pi_M(P)$ strictly decreases in P. Note that the asset buyers' equilibrium bid P_M^* satisfies $\Pi_M(P_M^*) = 0$, for $M \in \{1, 2\}$. In other words, the buyers always break even for a *realized* M and the associated belief $\omega_M^s(P_M^*)$. The regulator, on the other hand, can offer the price P_A^* to break even *across different numbers of runs* and thereby across possible posterior beliefs about the aggregate state s. To see this, we use the definition of $\Pi_M(P)$ to re-arrange $V_A(P_A)$ into the following form:

$$V_A(P_A) = \sum_{M=1}^{2} \left(\sum_{s=G,B} Pr(s) \cdot Pr(\theta < \theta^*(P_A)|s)^M \cdot Pr(\theta > \theta^*(P_A)|s)^{2-M} \right) C_2^M \cdot M \cdot \frac{D_1}{P_A} \cdot \Pi_M(P_A). \tag{25}$$

Importantly, the regulator does not require $\Pi_M(P_A^*) = 0$ but instead $V_A(P_A^*) = 0$. In fact, the regulator expects to make losses in the contingency of $M = 2$ and to profit in the contingency of $M = 1$. Since $\Pi_1(P_2^*) > 0$ and $\Pi_2(P_1^*) < 0$, we know that $V_A(P_A) > 0$ for $P_A = P_2^*$ and that $V_A(P_A) < 0$ for $P_A = P_1^*$. Since $V_A(P_A)$ monotonically and continuously decreases in P_A, there exists a unique $P_A^* \in (P_2^*, P_1^*)$ that allows the regulator to break even in expectation.

Figure 5: The equilibrium outcomes under the committed liquidity support



To appreciate the stability effect, suppose both banks' fundamentals are marginally below θ_2^*. We know that the equilibrium outcome in a laissez-faire market is that both banks fail and the prevailing asset price drops to P_2^*. Under the policy intervention, however, the systemic bank runs will no longer happen. Knowing that their banks can sell assets to the regulator at a pre-specified price $P_A^* > P_2^*$, the creditors can no longer rationalize x_2^* as their threshold signal. In this case, promoting financial stability does not involve the regulator purchasing banks' assets. The regulator improves the allocation by offering a support price to break down the feedback loop between contagious bank runs and asset sales. The regulator also reduces the risk of systemic bank runs when she actually purchases assets from the banks. In particular, the regulator will prevent systemic bank runs when one bank's cash flow is below $\theta_A^* = \theta^*(P_A^*)$ while the other's belongs to the interval $[\theta_A^*, \theta_2^*]$. The regulator saves the bank that has relatively strong fundamentals which would otherwise fail due to contagion. We illustrate the two cases by the cross-hatched and single-hatched areas in Figure 5, respectively.

4 Further Policy Discussion

We extend the policy discussion in Section 3.3 in three ways. We show that the regulator can induce banks' voluntary participation in the upfront liquidity arrangement (Section 4.1). We then analyze how the possibility for the regulator to bear losses from her intervention affects the effectiveness of the intervention (Section 4.2). We conclude the section by discussing whether the committed liquidity support remains a robust policy response to the dual-illiquidity problem when the regulator has information on the aggregate state s (Section 4.3).

4.1 Banks' voluntary participation in the upfront liquidity arrangement

Under the mutually committed liquidity support, it is assumed that banks commit to raising liquidity by selling their assets to the regulator even if private asset buyers may offer more. Interestingly, the regulator does not have to command banks to participate in the upfront liquidity arrangement at $t = 0$. We show that the regulator can induce banks' voluntary participation.

We analyze banks' incentives for voluntary participation with the following extension of our model. At $t = 0$, the regulator offers the two banks a price $P_A \in (P_2^*, P_1^*)$ at which it commits to purchasing banks' assets at $t = 1$ if runs happen. Banks' managements then decide whether to take the regulator's offer. Once a bank joins the arrangement, it is obliged to sell its assets only to the regulator (for example, assets are encumbered for this purpose). Otherwise, the bank sells its assets to private asset buyers. The two banks simultaneously decide whether to accept the regulator's offer. At $t = 1$, both banks' creditors and the private asset buyers observe P_A and whether a bank has joined the arrangement. For simplicity, a bank's management is assumed to receive a constant compensation b conditional on his bank being afloat. The management decides on whether to take the regulator's offer to maximize his expected compensation. The timing of the game is summarized in Figure 6.

We establish in Proposition 5 a sufficient condition under which the regulator can induce banks' voluntary participation. Intuitively, when one bank enters the agreement and receives the regulatory price support, the observation of a run on such a bank makes the private asset buyers particularly pessimistic about the aggregate state.[33] As a result, the bank that chooses not to participate in the program will face an even lower asset price when forced into liquidation.

[33]In terms of the policy intervention changing the informational environment, this result is related to Cong et al. (2020). In a dynamic global-games framework, the authors suggest that the initial policy intervention affects the informational environment of the subsequent interventions. In a different context, Szkup (2017) shows how a government can alter market participants' expectations via policy interventions to avoid a sovereign debt crisis.

Figure 6: Timing of the game with the regulator's offer and banks' decision

t = 0	t = 1	t = 2
1. Banks are established, with their portfolios and liability structures as given. 2. *The regulator offers a contract to buy bank assets at P_A in case any bank run happens.* 3. *The banks' managements simultaneously decide whether to accept the offer.*	1. State s and (θ^1, θ^2) are realized sequentially. 2. *Having observed P_A and banks' participation in the regulator's program*, creditors receive noisy signals and simultaneously decide on whether to run. 3. *When experiencing runs, a bank that participates in the regulator's program sells assets to the regulator at P_A; and a bank that refused to join the arrangement sells assets to the secondary-market asset buyers.* 4. When called to move, the secondary-market asset buyers bid for and acquire assets on sale.	1. Bank assets pay off. 2. Remaining obligations are settled.

This leads to an increased critical cash flow for the bank to survive and gives its management incentives to join the program in the first place. In other words, the bank that participates in the program exerts negative externalities on the bank that chooses not to; and both banks participating in the program can emerge as a Nash equilibrium.

Proposition 5. *There exists a unique critical $P_A^C \in (P_2^*, P_1^*)$, such that, when the regulator offers a price $P_A > P_A^C$, it is a Nash equilibrium for both banks to accept the offer. When $V_A(P_A^C) > 0$, we have $P_A^* > P_A^C$, and the regulator can induce voluntary participation with the price P_A^*.*

Proof. See Appendix B.6 □

While the regulator can induce commitment from the private institutions, the commitment from the regulator is essential. In fact, in a crisis, it is usually harder for a regulator to refrain from intervening than to provide rescues. The regulator's tendency to provide rescues is the driving mechanism of the collective moral hazard problem highlighted by Farhi and Tirole (2012). The tendency is empirically documented by Brown and Dinç (2011): the regulators tend to be accommodative with distressed banks when the banking sector faces the risk of a systemic crisis. The regulator can secure the commitment with legal obligations. For example, in providing the committed liquidity facility, the Reserve Bank of Australia enters legally binding agreements with participating banks that require the central bank to inject liquidity when fundings are needed. The regulator can also employ commitment devices, such as establishing financial stability funds, to signal her commitment to provide liquidity support.[34]

[34] A similar observation can be made about deposit insurance. Most countries require banks to pay deposit insurance premium ex-ante into a deposit insurance fund, which adds to the credibility of deposit insurance schemes.

Regulators can also be in positions to offer higher prices than private entities can do.[35] To start with, regulators have different objective functions as compared to private entities. Negative externalities from systemic bank failures are not taken into account by private asset buyers but are major concerns to regulators such as central banks. As we will show in the next section, a regulator may well want to avoid a systemic crisis at the cost of some monetary losses. Moreover, regulators such as central banks are not subject to the same stark bankruptcy constraints as private institutions and can sit on temporary losses. Similarly, central banks do not face pressure to lower the bid for banks' assets to increase the financial returns from their interventions, which may not be the case if the committed liquidity support is privately provided.

4.2 The possibility of public bail-outs

The policy intervention that we have analyzed so far assumes no public bailouts, in the sense that the regulator does not incur any expected loss and makes no net transfer to banks' claim holders. However, public authorities can make net losses in their interventions (e.g., see Laeven et al. (2010) for the fiscal cost of banking crises). Therefore, we relax this assumption to allow for (expected) losses from policy interventions. In particular, we replace the constraint (20) with $V_A(P_A) \geq -\overline{V}$, where $\overline{V} \geq 0$ indicates the regulator's loss-bearing capacity. We present the solution of the generalized regulator's program in Corollary 2.

Corollary 2. *When the regulator can bear a loss up to $\overline{V} \in (0, -V_A(qD_2))$, she optimally commits to purchasing banks' assets at a price $P_A^{**} > P_A^*$ to reduce the risk of systemic failures from $SYS(P_A^*)$ to $SYS(P_A^{**})$. When the regulator's loss-bear capacity is sufficiently large, $\overline{V} \geq -V_A(qD_2)$, the regulator will offer a price $P_A = qD_2 \geq P_A^{**}$ to eliminate funding liquidity risks.*

Proof. See Appendix B.7 □

The intuition of the result is as follows: since both the probability of systemic failures $\text{SYS}(P_A)$ and the regulator's expected payoff $V_A(P_A)$ decrease in the regulator's offer P_A, the regulator can commit to a higher price P_A^{**} such that $V_A(P_A^{**}) = -\overline{V}$ once she is allowed to make a loss $\overline{V}$. However, when the loss-bearing capacity $\overline{V}$ exceeds the cutoff $-V_A(qD_2)$, the constraint (21) becomes binding. The regulator will not raise her offer above qD_2 to avoid saving insolvent banks.

[35] Since the party that offers the price support does not make expected losses, it can be, in principle, a private institution, or more realistically, a large number of private institutions that participate in a decentralized market to provide liquidity insurance, in which case the commitments must be due to contracts enforceable by a court.

The possibility for the regulator to take expected losses also broadens the set of policy tools that can be used to promote financial stability. In particular, it is now possible for the regulator to unilaterally commit to purchasing banks' assets for a price $P_U \geq P_2^*$ independently of the number of runs.

When $P_U \geq P_1^*$, creditors expect banks to sell their assets only to the regulator when experiencing runs. The analysis resembles that in Section 3.3.2. Complications arise for $P_U \in [P_2^*, P_1^*)$. A creditor now expects his bank to sell its assets to the regulator at $P_U \geq P_2^*$ only when both banks experience runs. In contrast, the creditor expects his bank to sell its assets to the private asset buyers for $P_1^* > P_U^*$ when his bank is the only one that is forced into liquidation. The unilaterally committed liquidity support, therefore, can co-exist with the private secondary market for banks' assets. It can be shown that creditors in a bank i have an equilibrium strategy to withdraw if and only if their private signals fall below $x^*(y^{-i}, P_U)$, with[36]

$$x^*(y^{-i}, P_U) = \begin{cases} x^*(P_U) & y^{-i} < x^*(P_U) + \eta + \epsilon \\ x_1^* & y^{-i} \geq x^*(P_U) + \eta + \epsilon. \end{cases}$$

When $\eta \to 0$ and $\epsilon \to 0$, we have $x^*(P_U) \to \theta^*(P_U) = (D_2 - D_1)/(1 - qD_1/P_U)$. Under the intervention, a systemic banking crisis happens if and only if both banks' cash flows fall below $\theta^*(P_U)$. When the regulator unilaterally commits to purchasing banks' assets at a price P_U, the risk of systemic bank failures becomes $\text{SYS}(P_U) = \alpha \cdot \left(\frac{\theta^*(P_U) - \underline{\theta}_G}{\bar{\theta} - \underline{\theta}_G}\right)^2 + (1-\alpha) \cdot \left(\frac{\theta^*(P_U) - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right)^2$.

When offering a $P_U \in [P_2^*, P_1^*)$, the regulator expects to purchase banks' assets only when both banks experience runs, in which case she earns an expected payoff

$$V_U(P_U) = \sum_{s=G,B} Pr(s)\left(Pr(\theta < \theta^*(P_U)|s)^2 \cdot 2 \cdot \frac{D_1}{P_U} \cdot \pi(P_U|s)\right) \quad \text{for} \quad P_U \in [P_2^*, P_1^*). \tag{26}$$

Instead, when $P_U \geq P_1^*$, the banks would only trade with the regulator so that the regulator's expected payoff is identical to the expression (24):

$$V_U(P_U) = V_A(P_U) \quad \text{for} \quad P_U \geq P_1^*. \tag{27}$$

The regulator chooses a $P_U \geq P_2^*$ to minimize $\text{SYS}(P_U)$, subject to the constraints $V_U(P_U) \geq -\overline{V}$ and $\theta^*(P_U) \geq D_2$. By unilaterally committing to a price $P_U \in [P_2^*, P_1^*)$, the regulator expects to purchase assets only when runs happen to both banks, in which case, any price higher

[36]The solution of the game resembles that in Section 3.2.2. We provide detailed proof in the Online Appendix.

than P_2^* implies expected losses. When the regulator can bear a moderate amount of losses $\overline{V} \in (0, -V_U(qD_2))$, she would exhaust her loss-bearing capacity to minimize the systemic risk. That is, she optimally commits to purchasing banks' assets for a price P_U^{**} so that $V_U(P_U^{**}) = -\overline{V}$. When the regulator's loss-bearing capacity is above $-V_U(qD_2)$, the constraint $\theta^*(P_U) \geq D_2$ becomes binding — the regulator sets her price support at qD_2 to minimize the systemic risk while avoiding saving insolvent banks. Lemma 4 summarizes the results.

Lemma 4. *If the regulator unilaterally commits to purchasing banks' assets on sale at $t = 1$, the only price that allows her to break even is P_2^*. If the regulator can take an expected loss up to $\overline{V} \in (0, -V_U(qD_2))$, she will commit to a price $P_U^{**} > P_2^*$ with P_U^{**} satisfying $V_U(P_U^{**}) = -\overline{V}$. If the regulator can take an expected loss $\overline{V} \geq -V_U(qD_2)$, she will offer a price equal to qD_2.*

Proof. See Appendix B.8. □

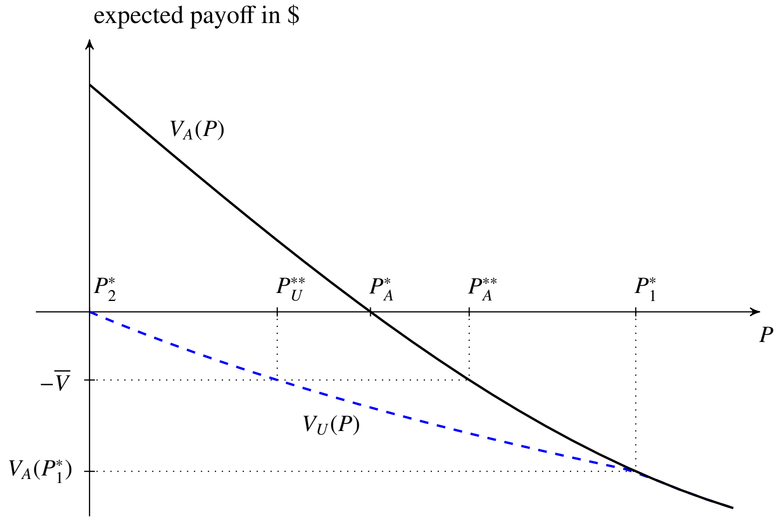
Given a loss-bearing capacity $\overline{V} \geq 0$, we now compare the stability effects of an arrangement with the regulator's unilateral commitment and that with the mutual commitments from the regulator and banks. Since the regulator aims to minimize the systemic risk, an arrangement will achieve greater stability if it allows the regulator to offer a higher price for banks' assets. We establish in Proposition 6 that the ex-ante liquidity support with the mutual commitments dominates that with the regulator's unilateral commitment. Figure 7 illustrates the comparison.

Proposition 6. *When the regulator has a loss-bearing capacity $\overline{V} < -V_A(P_1^*) = -V_U(P_1^*)$, an upfront liquidity arrangement with the mutual commitments outperforms that with the regulator's unilateral commitment in terms of containing the risk of systemic bank failures. The two arrangements are equally effective otherwise.*

When the regulator cannot make any expected loss (i.e., $\overline{V} = 0$), the ex-ante liquidity support with the mutual commitments strictly dominates that with the regulator's unilateral commitment, since the regulator can only offer P_2^* under the latter arrangement. The former arrangement also outperforms the latter when the regulator has a loss-bearing capacity $\overline{V} \in (0, -V_A(P_1^*))$. For such a moderate loss-bearing capacity, the regulator can offer a price $P_A^{**} \in (P_A^*, P_1^*)$ under the mutual commitments, while she can only offer a lower price $P_U^{**} \in (P_2^*, P_1^*)$ with her unilateral commitment. To show $P_A^{**} > P_U^{**}$, observe that for a price $P \in (P_2^*, P_1^*)$, the regulator's expected payoffs under the two arrangements are linked as follows:

$$V_A(P) = V_U(P) + \left(\sum_{s=G,B} Pr(s) \cdot Pr(\theta < \theta^*(P)|s) \cdot Pr(\theta > \theta^*(P)|s) \cdot 2 \cdot \frac{D_1}{P} \right) \cdot \Pi_1(P). \tag{28}$$

Figure 7: The regulator's payoffs under mutual vs. unilateral commitments



We plot the regulator's expected payoff from purchasing banks' assets at a price P with the mutual commitments $V_A(P)$, and that with her unilateral commitment $V_U(P)$. The regulator can offer $P_A^* \in (P_2^*, P_1^*)$ under the mutual commitments to break even, while the only price that allows her to break even under her unilateral commitment is P_2^*. For a loss-bearing capacity $\overline{V} \in (0, -V_A(P_1^*))$, the regulator optimally offers $P_A^{**} \in (P_A^*, P_1^*)$ under the mutual commitments and offers $P_U^{**} < P_A^{**}$ under her unilateral commitment. The regulator will offer the same price under both arrangements when her loss-bearing capacity is greater than $-V_A(P_1^*)$.

Note that $V_A(P) > V_U(P)$ holds for any $P \in [P_2^*, P_1^*)$ since $\Pi_1(P) > 0$.[37] In particular, since $P_U^{**} \in (P_2^*, P_1^*)$, we have $V_A(P_U^{**}) > V_U(P_U^{**}) = -\overline{V}$. By the monotonicity of $V_A(P)$, we obtain $P_A^{**} > P_U^{**}$ when $\overline{V} \in (0, -V_A(P_1^*))$. When the regulator can make large losses, i.e., $\overline{V} > -V_A(P_1^*)$, she would receive the same expected payoff under both arrangements when committing to a price greater than P_1^*. Consequently, the two arrangements are equally effective in this case.

We believe it is both reasonable and realistic to demand commitment from banks. From a normative point of view, the public liquidity support that banks can access should be commensurate with their regulatory obligations. Banks should take regulatory obligations (e.g., entering a binding agreement with the regulator) in exchange for public liquidity support during crisis times. Furthermore, it is feasible for banks to make such commitments. For instance, banks can put assets in encumbrance and reserve these assets exclusively for raising liquidity from central banks. Alternatively, banks can pay for public liquidity support upfront. For example, the committed liquidity facility of the Reserve Bank of Australia requires banks participating in the program to make ex-ante payments for the central bank's liquidity insurance.

[37] Recall that $\Pi_1(P_1^*) = 0$ and $\Pi_1(P)$ monotonically decreases in P.

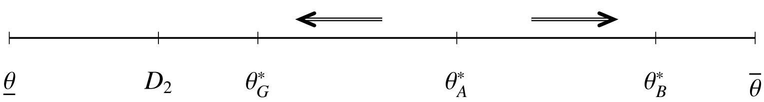
4.3 Policy interventions with knowledge on the aggregate state

Since the financial contagion in our model is driven by the incomplete information on the aggregate state, it can be particularly relevant to assess whether the committed liquidity support remains a robust policy option when the state s can be learned. Indeed, if s is interpreted as a macro-economic variable, it is not unreasonable to assume that a regulator is better informed than the rest of the economy.[38] If the regulator can disclose the aggregate state or purchase banks' assets after learning s, the asset prices would be conditional on the aggregate state: i.e., P_B^* and P_G^* in state B and G respectively. The asset prices will generate critical fundamentals θ_B^* and θ_G^*. Such a policy creates the following trade-off. When $s = G$, the policy boosts the asset price and saves banks with $\theta \in (\theta_G^*, \theta_A^*)$ from illiquidity. However, when $s = B$, the policy exacerbates liquidity problems by pushing the critical cash flow to θ_B^*. We depict the trade-off in Figure 8. It should be clear that interventions conditional on s do not necessarily dominate the committed liquidity support. We summarize the result in Proposition 7.

Proposition 7. *Suppose that the regulator knows the realized aggregate state $s \in \{G, B\}$. There exists a $\overline{\theta}^c$ such that, for $\overline{\theta} > \overline{\theta}^c$, the committed liquidity support with the price P_A^* dominates interventions conditional on s, such as disclosing s or purchasing banks' assets for price P_s^*.*

Proof. See Appendix B.9. □

Figure 8: Intervention conditional on the aggregate state s



5 Concluding remarks

In this paper, we revisit a classic issue of liquidity support for troubled banks. We emphasize that both private investors and central banks can face the information constraint that it can be difficult — if not impossible — to distinguish illiquid banks from the insolvent ones

[38] Also in a global-games framework, Eisenbach (2017) assumes observable aggregate states and suggests using contingent liabilities to maintain both financial stability and the disciplinary power of runs. By contrast, as we will show in this section, the committed liquidity support designed without the information on the realized aggregate state s can still outperform policies that are based on more precise information.

in crisis times. We introduce such an information constraint into a global-games framework where the solvent-but-illiquid banks are endogenously defined. We endogenize the liquidation value of banks' assets under the information constraint and show how a bank's funding illiquidity interacts with its asset illiquidity. In a two-bank setting with aggregate uncertainties, a vicious cycle emerges between contagious bank runs and falling asset prices. We analyze a global games model with multiple groups of players and multi-dimensional signals, so that we obtain a unique equilibrium for clear-cut policy analysis despite the two-way feedback between collapsing asset prices and contagious bank runs.

Our model illustrates how the lack of precise information on banks' solvency creates financial fragility and simultaneously restricts the set of feasible policy tools: without granular information on individual banks' solvency, it is infeasible for central banks to target only solvent-but-illiquid banks as suggested by Bagehot's principles. Instead, we show that a regulator with information on neither the individual banks' solvency nor the aggregate risk factor can break down the two-way feedback between failing asset prices and contagious bank runs with an upfront liquidity support. In particular, we recommend an arrangement where a regulator and banks mutually commit to an agreement for the regulator to purchase a bank's assets for a pre-specified price to contain contagious bank runs. The arrangement reduces the risk of systemic bank failures while allowing a central bank to rescue no insolvent banks and to make no loss in the intervention — even if the central bank possesses no information on banks' solvency. Our theory on committed liquidity support helps rationalize some recent policy practices, such as the asset prepositioning program of the Bank of England and the committed liquidity facility of the Reserve Bank of Australia.

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Appendix A Preliminaries of the bank run game

Appendix A.1 Upper and Lower dominance regions

We first show that $\theta^L \equiv D_2$ defines a lower dominance region $[\underline{\theta}_s, \theta^L)$, where it is a creditor's dominant strategy to withdraw early — independent of the other creditors' actions and for any asset price $P \in [\underline{P}, qD_2)$. Indeed, when the bank is fundamentally insolvent (i.e., $\theta^i < D_2$), the inequality $(1-\lambda^i)\theta^i < F + (1-L^i)(1-E-F)r_D$ will hold for $L^i = 0$, and the creditor will always be better off to withdraw than to wait. He will receive a zero payoff by choosing to wait because of the bank failure, but will receive D_1 if he withdraws early.[39] Therefore, if a creditor's private signal falls below $D_2 - \epsilon$, he is sure that $\theta < D_2$ and his dominant strategy is to withdraw.

Similarly, $\theta^U \equiv F/(1 - D_1/\underline{P})$ defines an upper dominance region $(\theta^U, \overline{\theta}]$, where it is a creditor's dominant strategy to wait – independent of other creditors' actions and for any asset price $P \in [\underline{P}, qD_2)$. To see this, suppose that all other creditors withdraw early (i.e., $L^i = 1$) and that the asset price is the least favorable and equal to $\underline{P}$. The bank can still repay its liabilities in full if its fundamentals exceed $\theta^U(\underline{P})$. Therefore, the creditor will receive D_1 if he withdraws, and D_1/q if he waits. Since $\theta^U(\cdot)$ decreases in the asset price P, "wait" remains a dominant strategy for any price $P > \underline{P}$. Provided that $\overline{\theta} > F/(1 - D_1/\underline{P})$, the upper dominance region exists. When a creditor's private signal exceeds $\theta^U(\underline{P}) + \epsilon$, he will be sure that $\theta > \theta^U(\underline{P})$ and his dominant strategy is to wait.

Appendix A.2 The fraction of withdrawals in the bank i and the bank $-i$

In this section, we derive a bank i's representative creditor j's rational expectations on the total withdrawals L^i in the bank i and L^{-i} in the bank $-i$ — given that all other creditors take the equilibrium threshold strategy and that the creditor observes signals (x^i_j, y^{-i}).

The fraction of early withdrawals in the bank i

Observing an outside signal y^{-i} about θ^{-i}, the representative creditor knows that all other creditors in the bank i withdraw if and only if their private signals are below $x^*(y^{-i})$. L^i is a function of the bank's fundamentals θ^i and the critical signal $x^*(y^{-i})$, i.e., $L^i(\theta^i, x^*(y^{-i}))$.

[39] Recall from Lemma 1 that banks will not fail at $t = 1$, since an equilibrium asset price must be higher than $\underline{P}$ which is further higher than D_1.

We derive the function form of $L^i(\theta^i, x^*(y^{-i}))$. For a realized θ^i, there can be three cases. (i) When $\theta^i < x^*(y^{-i}) - \epsilon$, the highest possible private signal is still below $x^*(y^{-i})$. All creditors withdraw early in the bank i and $L^i(\theta^i, x^*(y^{-i})) = 1$. (ii) When $\theta^i > x^*(y^{-i}) + \epsilon$, the lowest possible private signal still exceeds $x^*(y^{-i})$. All creditors wait in the bank i and $L(\theta^i, x^*(y^{-i})) = 0$. (iii) For $\theta^i \in [x^*(y^{-i}) - \epsilon, x^*(y^{-i}) + \epsilon]$, we denote by x^i_k the private signal of another creditor k of the bank i. In this case, $L^i(\theta^i, x^*(y^{-i}))$ can be calculated as follows.

$$L^i(\theta^i, x^*(y^{-i})) = Prob\left(x^i_k < x^*(y^{-i})\middle|\theta^i, y^{-i}\right) = Prob\left(\epsilon^i_k < x^*(y^{-i}) - \theta^i\middle|\theta^i, y^{-i}\right) = \frac{x^*(y^{-i}) - \theta^i + \epsilon}{2\epsilon} \quad \text{(A.29)}$$

To summarize, $L^i(\theta^i, x^*(y^{-i}))$ can be expressed as follows:

$$L^i(\theta^i, x^*(y^{-i})) = \max\left\{\min\left\{\frac{x^*(y^{-i}) - \theta^i + \epsilon}{2\epsilon}, 1\right\}, 0\right\} = \begin{cases} 1 & \theta^i < x^*(y^{-i}) - \epsilon \\ \frac{x^*(y^{-i}) - \theta^i + \epsilon}{2\epsilon} & x^*(y^{-i}) - \epsilon \le \theta^i \le x^*(y^{-i}) + \epsilon \\ 0 & \theta^i > x^*(y^{-i}) + \epsilon. \end{cases} \quad \text{(A.30)}$$

The representative creditor j perceives $L^i(\theta^i, x^*(y^{-i}))$ uncertain since he only receives a noisy signal $x^i_j = \theta^i + \epsilon^i_j$. In particular, he will form a posterior belief $\theta^i|_{x^i_j} \sim U(x^i_j - \epsilon, x^i_j + \epsilon)$ and rationally expect the total withdrawals $L^i(x^i_j, y^{-i})$ in the bank i to be

$$L^i(x^i_j, y^{-i}) = \int_{x^i_j - \epsilon}^{x^i_j + \epsilon} L^i(\theta^i, x^*(y^{-i})) \cdot \frac{1}{2\epsilon} \cdot d\theta^i = E\left[L^i(\theta^i, x^*(y^{-i}))\middle|x^i_j, y^{-i}\right]. \quad \text{(A.31)}$$

It can be verified that $L^i(x^i_j, y^{-i})$ has the following functional form: $L^i(x^i_j, y^{-i}) = 0$ when $x^i_j > x^*(y^{-i}) + 2\epsilon$, $L^i(x^i_j, y^{-i}) = [x^*(y^{-i}) - x^i_j + 2\epsilon]^2/(8\epsilon^2)$ when $x^*(y^{-i}) < x^i_j \le x^*(y^{-i}) + 2\epsilon$, $L^i(x^i_j, y^{-i}) = 1/2$ when $x^i_j = x^*(y^{-i})$, $L^i(x^i_j, y^{-i}) = 1/2 + [x^*(y^{-i}) - x^i_j]/(2\epsilon) - [x^*(y^{-i}) - x^i_j]^2/(8\epsilon^2)$ when $x^*(y^{-i}) - 2\epsilon \le x^i_j < x^*(y^{-i})$, and $L^i(x^i_j, y^{-i}) = 1$ when $x_j < x^*(y^{-i}) - 2\epsilon$.

The fraction of early withdrawals in the bank $-i$

Like the previous case, L^{-i} is a function of the bank $-i$'s fundamentals θ^{-i} and the bank $-i$'s creditors' critical signal $x^*(y^i)$. Note that $y^i = \theta^i + \eta^i$ is those creditors' outside signal about the bank i's fundamentals. Following the same step, we have

$$L^{-i}(\theta^{-i}, x^*(y^i)) = \max\left\{\min\left\{\frac{x^*(y^i) - \theta^{-i} + \epsilon}{2\epsilon}, 1\right\}, 0\right\} = \begin{cases} 0 & \theta^{-i} > x^*(y^i) + \epsilon \\ \frac{x^*(y^i) - \theta^{-i} + \epsilon}{2\epsilon} & x^*(y^i) - \epsilon \le \theta^{-i} \le x^*(y^i) + \epsilon \\ 1 & \theta^{-i} < x^*(y^i) - \epsilon. \end{cases} \quad \text{(A.32)}$$

Observing a noisy signal $y^{-i} = \theta^{-i} + \eta^{-i}$, the representative creditor j in the bank i perceives θ^{-i} uncertain. Besides this uncertainty about θ^{-i}, the representative creditor j also perceives the critical signal $x^*(y^i)$ in the other bank uncertain: he considers the outside signal y^i observed by the bank $-i$'s creditors a random variable Y^i.

To derive the representative creditor j's rational expectations on L^{-i}, we first calculate his posterior belief about Y^i conditional on the private signal x_j^i, i.e., the conditional distribution $F_{Y^i}(y^i|x_j^i)$ and the conditional density $f_{Y^i}(y^i|x_j^i)$. Since $\eta^i \sim U(-\eta, \eta)$, we have $Prob(Y^i \leq y^i|\theta^i) = Prob(\theta^i + \eta^i \leq y^i|\theta^i) = Prob(\eta^i \leq y^i - \theta^i)$. For a realized θ^i, the distribution of the outside signal Y^i can be expressed as follows:

$$Prob(Y^i \leq y^i|\theta^i) = \max\left\{\min\left\{\frac{y^i - \theta^i + \eta}{2\eta}, 1\right\}, 0\right\} = \begin{cases} 0 & \theta^i > y^i + \eta \\ \frac{y^i - \theta^i + \eta}{2\eta} & y^i - \eta \leq \theta^i \leq y^i + \eta \\ 1 & \theta^i < y^i - \eta. \end{cases}$$

In this expression, y^i can be any real number. The representative creditor j perceives $Prob(Y^i \leq y^i|\theta^i)$ uncertain since he only receives x_j^i about θ^i. We have:

$$Prob\left(Y^i \leq y^i \middle| x_j^i\right) = \int_{x_j^i - \epsilon}^{x_j^i + \epsilon} \max\left\{\min\left\{\frac{y^i - \theta^i + \eta}{2\eta}, 1\right\}, 0\right\} \cdot \frac{1}{2\epsilon} \cdot d\theta^i.$$

It can be be verified that $Prob\left(Y^i \leq y^i \middle| x_j^i\right)$ has the following functional form:

$$F_{Y^i}(y^i|x_j^i) = Prob(Y^i \leq y^i|x_j^i) = \begin{cases} 0 & y^i \leq x_j^i - \epsilon - \eta \\ \frac{\left[(y^i+\eta)-(x_j^i-\epsilon)\right]^2}{8\eta\epsilon} & x_j^i - \epsilon - \eta < y^i \leq x_j^i + \epsilon - \eta \\ \frac{y^i+\eta-x_j^i}{2\eta} & x_j^i + \epsilon - \eta < y^i \leq x_j^i - \epsilon + \eta \\ 1 - \frac{\left[(y^i-\eta)-(x_j^i+\epsilon)\right]^2}{8\eta\epsilon} & x_j^i - \epsilon + \eta < y^i \leq x_j^i + \epsilon + \eta \\ 1 & y^i > x_j^i + \epsilon + \eta. \end{cases} \tag{A.33}$$

Take the first order derivative with respect to y^i, we obtain the conditional density function.

$$f_{Y^i}(y^i|x_j^i) = \begin{cases} \frac{y^i-(x_j^i-\eta-\epsilon)}{4\eta\epsilon} & x_j^i - \eta - \epsilon < y^i \leq x_j^i - \eta + \epsilon \\ \frac{1}{2\eta} & x_j^i - \eta + \epsilon < y^i \leq x_j^i + \eta - \epsilon \\ \frac{(x_j^i+\eta+\epsilon)-y^i}{4\eta\epsilon} & x_j^i + \eta - \epsilon < y^i \leq x_j^i + \eta + \epsilon \\ 0 & \text{otherwise.} \end{cases} \tag{A.34}$$

Note that $f_{Y^i}(y^i|x_j^i)$ is everywhere non-negative, and it is strictly positive when $x_j^i - \eta - \epsilon < y^i \leq x_j^i + \eta + \epsilon$.

Second, based on the outside signal y^{-i}, the representative creditor j will form a posterior belief $\theta^{-i}|_{y^{-i}} \sim U(y^{-i} - \eta, y^{-i} + \eta)$ about the bank $-i$'s fundamentals θ^{-i}.

Lastly, $\theta^{-i}|_{y^{-i}}$ and $y^i|_{x_j^i}$ are independently distributed since the noises ϵ_j^i, η^i and η^{-i} are independently distributed. The joint density function of $\theta^{-i}|_{y^{-i}}$ and $y^i|_{x_j^i}$ is $\frac{1}{2\eta} \cdot f_{Y^i}(y^i|x_j^i)$. Apply Fubini's theorem, the representative creditor j's expectations on the total withdrawals $L^{-i}(x_j^i, y^{-i})$ in the bank $-i$ can be expressed as follows:

$$L^{-i}(x_j^i, y^{-i}) = E\left\{E\left[L^{-i}(\theta^{-i}, x^*(y^i))\middle|y^{-i}\right]\middle|x_j^i\right\} = \int_{x_j^i-\eta-\epsilon}^{x_j^i+\eta+\epsilon}\int_{y^{-i}-\eta}^{y^{-i}+\eta} L^{-i}(\theta^{-i}, x^*(y^i)) \cdot \frac{1}{2\eta} \cdot d\theta^{-i} \cdot f_{Y^i}(y^i|x_j^i) \cdot dy^i. \quad \text{(A.35)}$$

The expressions of $L^{-i}(\theta^{-i}, x^*(y^i))$ and $f_{Y^i}(y^i|x_j^i)$ are given by (A.32) and (A.34), respectively.

Appendix A.3 Monotonicity of $L^i(x_j^i, y^{-i})$ and $L^{-i}(x_j^i, y^{-i})$

The following Result 1 establishes the monotonicity of $L^i(x_j^i, y^{-i})$ and $L^{-i}(x_j^i, y^{-i})$.

Result 1. *We have* $\frac{\partial L^i(x_j^i,y^{-i})}{\partial x_j^i} \le 0$, $\frac{\partial L^i(x_j^i,y^{-i})}{\partial y^{-i}} \le 0$, $\frac{\partial L^{-i}(x_j^i,y^{-i})}{\partial x_j^i} \le 0$ *and* $\frac{\partial L^{-i}(x_j^i,y^{-i})}{\partial y^{-i}} \le 0$.

Proof. Recall the functional form of $L^i(x_j^i, y^{-i})$. We have $\partial L^i(x_j^i, y^{-i})/\partial x_j^i = -[x^*(y^{-i}) - x_j^i + 2\epsilon]/(4\epsilon^2) < 0$ when $x_j^i \in (x^*(y^{-i}), x^*(y^{-i}) + 2\epsilon)$, $\partial L^i(x_j^i, y^{-i})/\partial x_j^i = -[x_j^i - x^*(y^{-i}) + 2\epsilon]/(4\epsilon^2) < 0$ when $x_j^i \in (x^*(y^{-i}) - 2\epsilon, x^*(y^{-i})]$, and $\partial L^i(x_j^i, y^{-i})/\partial x_j^i = 0$ otherwise. So we have $\partial L^i(x_j^i, y^{-i})/\partial x_j^i \le 0$. Similarly, we have $\partial L^i(x_j^i, y^{-i})/\partial x^*(y^{-i}) = [x^*(y^{-i}) - x_j^i + 2\epsilon]/(4\epsilon^2) > 0$ when $x_j^i \in (x^*(y^{-i}), x^*(y^{-i}) + 2\epsilon)$, $\partial L^i(x_j^i, y^{-i})/\partial x^*(y^{-i}) = [x_j^i - x^*(y^{-i}) + 2\epsilon]/(4\epsilon^2) > 0$ when $x_j^i \in (x^*(y^{-i}) - 2\epsilon, x^*(y^{-i})]$, and $\partial L^i(x_j^i, y^{-i})/\partial x^*(y^{-i}) = 0$ otherwise. Provided that $x^*(y^{-i})$ is monotonically decreasing in y^{-i}, we have $\partial L^i(x_j^i, y^{-i})/\partial y^{-i} \le 0$.

Recall $F_{Y^i}(y^i|x_j^i)$ in the expression (A.33). It can be verified that $F_{Y^i}(y^i|x')$ first order stochastically dominates $F_{Y^i}(y^i|x)$ for any $x' > x$, i.e., the conditional distribution of Y^i "shifts left" when x_j^i becomes lower. Provided that $x^*(y^i)$ is monotonically decreasing in y^i, the representative creditor j believes that it is more likely for the bank $-i$'s creditors to follow a higher threshold signal $x^*(y^i)$ when observing a lower x_j^i. Combined with the fact that $L^{-i}(\theta^{-i}, x^*(y^i))$ monotonically increases in $x^*(y^i)$, we have $\partial L^{-i}(x_j^i, y^{-i})/\partial x_j^i \le 0$. Lastly, in the expression (A.35), one can directly calculate the following:

$$\begin{aligned}
&\frac{\partial}{\partial y^{-i}}\left(\int_{y^{-i}-\eta}^{y^{-i}+\eta} \max\left\{\min\left\{\frac{x^*(y^i) - \theta^{-i} + \epsilon}{2\epsilon}, 1\right\}, 0\right\} \cdot d\theta^{-i}\right) \\
&= \max\left\{\min\left\{\frac{x^*(y^i) - y^{-i} - \eta + \epsilon}{2\epsilon}, 1\right\}, 0\right\} - \max\left\{\min\left\{\frac{x^*(y^i) - y^{-i} + \eta + \epsilon}{2\epsilon}, 1\right\}, 0\right\} \le 0.
\end{aligned}$$

We have $\partial L^{-i}(x_j^i, y^{-i})/\partial y^{-i} \le 0$ since the conditional density $f_{Y^i}(y^i|x_j^i)$ is everywhere non-negative. □

Appendix B Proofs to lemmas and propositions

Appendix B.1 Proof of Lemma 1

Proof. Since buyers' bid cannot be negative, an ex-post break-even price P after observing M runs, $M \in \{1, 2\}$, if exits, must be in one of the three regions: $[0, \underline{P})$, $[\underline{P}, qD_2)$, or $[qD_2, +\infty)$. We show that it cannot be greater than or equal to qD_2, nor can it be lower than $\underline{P} = (\underline{\theta}_B + D_2)/2$.

Suppose $P \geq qD_2$, then it is not sequentially rational for the wholesale creditors to withdraw from a solvent bank, i.e., $\theta^i > D_2$. To see this, one can take the perspective of a representative creditor j of a bank i. Even when all other creditors withdraw, the bank needs to liquidate no more than D_1/qD_2 fraction of its asset, for $P \geq qD_2$. While the bank's $t = 2$ liability drops to F, its residual cash flow is $(1 - D_1/P) \cdot \theta^i \geq [1 - D_1/(qD_2)]D_2 \geq F$ as $\theta^i \geq D_2$. As a result, by running on the bank, creditor j will only incur a penalty for early withdrawal. This implies that whenever a run happens when $P \geq qD_2$, the bank must be fundamentally insolvent with $\theta^i < D_2$. Therefore, buyers must expect asset quality to be lower than $(D_2 + \underline{\theta}_G)/2$, which is in turn lower than qD_2 given our parametric assumption (3). Buyers would make a loss by offering $P \geq qD_2$, a contradiction.

A break-even price P cannot be smaller than $\underline{P}$ either. Note that when a bank is fundamentally insolvent with cash flow $\theta^i < D_2$, it is a dominant strategy for its wholesale creditors to run independently of the asset price. To see this, notice that if $P \geq D_1$ and the bank does not fail at $t = 1$, a creditor is better off to run and receive D_1 than to wait and receive 0.[40] On the other hand, if $P < D_1$, a creditor will receive a zero payoff for his claim whether he runs or not, but can still obtain an arbitrarily small reputational benefit by running on a bank that is doomed to fail. This implies that runs must happen to those banks with $\theta^i < D_2$, and the expected quality of assets on sale is at least $(\underline{\theta}_B + D_2)/2 = \underline{P}$. As asset buyers break even with their competitive bidding, the price they offer must be greater than or equal to $\underline{P}$. □

Appendix B.2 Proof of Proposition 1

Proof. To start with, we present a bank i's representative creditor j's payoff difference function and derive his best response to other players' equilibrium strategy, i.e., $x^*(\cdot)$ and P^*. Rationally expecting a price P^* independent of the runs, the representative creditor's payoff difference $E\left[DW(L^i(x^i_j, y^{-i}), \theta^*(y^{-i}), P^*)\right]$ can be expressed as follows:

[40] Note that the ex-post asset sale will never revive an insolvent bank as we prove that $P \geq qD_2$ could never happen.

$$E\left[DW(L^i(x^i_j, y^{-i}), \theta^*(y^{-i}), P^*)\right] = \begin{cases} \frac{1-q}{q} D_1 & x^i_j > \overline{x} \\ \frac{D_1}{q}[L^c(\theta^*(y^{-i}), P^*) - q] + \frac{D_1}{q}\frac{x^i_j - x^*(y^{-i})}{2\epsilon} & x^i_j \in (x^*(y^{-i}), \overline{x}] \\ \frac{D_1}{q}[L^c(\theta^*(y^{-i}), P^*) - q] & x^i_j = x^*(y^{-i}) \\ \frac{D_1}{q}[L^c(\theta^*(y^{-i}), P^*) - q] - \frac{D_1}{q}\frac{x^*(y^{-i}) - x^i_j}{2\epsilon} & x^i_j \in [\underline{x}, x^*(y^{-i})) \\ -D_1 & x^i_j < \underline{x}. \end{cases} \tag{B.36}$$

In (B.36), $\overline{x} \in (x^*(y^{-i}), x^*(y^{-i}) + 2\epsilon]$ and $\underline{x} \in [x^*(y^{-i}) - 2\epsilon, x^*(y^{-i}))$ are two cutoffs of x^i_j, which solve $L^c(\theta^*(y^{-i}), P^*) = 1 - (\overline{x} - x^*(y^{-i}))/2\epsilon$ and $(x^*(y^{-i}) - \underline{x})/2\epsilon = L^c(\theta^*(y^{-i}), P^*)$ respectively.[41] In addition, the expression of $L^c(\theta^*(y^{-i}), P^*)$ is given in (13).

One can check the following results. First, the payoff difference function is linear in x^i_j with a slope $D_1/(2q\epsilon) > 0$ when $x^i_j \in (\underline{x}, \overline{x}) \subset [x^*(y^{-i}) - 2\epsilon, x^*(y^{-i}) + 2\epsilon]$. Second, the payoff difference function equals a constant $(1-q)D_1/q > 0$ when $x^i_j \geq \overline{x}$ and another constant $-D_1 < 0$ when $x^i_j \leq \underline{x}$. Therefore, there must exist a unique $\hat{x} \in (\underline{x}, \overline{x})$ such that $E\left[DW(L^i(\hat{x}, y^{-i}), \theta^*(y^{-i}), P^*)\right] = 0$, $\forall y^{-i} \in [\underline{\theta}_s - \eta, \overline{\theta} + \eta]$. This establishes a function $\hat{x}(y^{-i})$, $\forall y^{-i} \in [\underline{\theta}_s - \eta, \overline{\theta} + \eta]$. The representative creditor j's best response to the other creditors' threshold strategy (i.e., $x^*(\cdot)$) is a threshold strategy: to withdraw if $x^i_j < \hat{x}(y^{-i})$ and to wait if $x^i_j > \hat{x}(y^{-i})$, $\forall y^{-i} \in [\underline{\theta}_s - \eta, \overline{\theta} + \eta]$. We have

$$\hat{x}(y^{-i}) = x^*(y^{-i}) - 2\epsilon\left[L^c\left(\theta^*(y^{-i}), P^*\right) - q\right],$$

by solving $E\left[DW(L^i(\hat{x}, y^{-i}), \theta^*(y^{-i}), P^*)\right] = 0$. In a symmetric equilibrium, $\hat{x}(y^{-i}) = x^*(y^{-i})$ must be true. Therefore, the critical cash flow must satisfy $L^c(\theta^*(y^{-i}), P^*) = q$, which results in the equilibrium condition (14) in the text.

Following the equilibrium conditions (15) and (16), the critical cash flow $\theta^*(y^{-i})$ and the critical signal $x^*(y^{-i})$, if exist, are constants and do not depend on y^{-i}. So we denote them as θ^* and $x^* = \theta^* + (2q-1)\epsilon$. The rest of the proof then shows the existence and uniqueness of a combination (θ^*, x^*, P^*) jointly solving the following system of equations.

$$\theta^* = \frac{D_2 - D_1}{1 - qD_1/P^*}, \quad x^* = \theta^* + (2q-1)\epsilon, \quad P^* = \frac{\theta^* + \underline{\theta}}{2}.$$

Insert the expression of θ^* into that of P^* and define a function $\Pi(P)$ as the asset buyers' expected profit function from purchasing banks' assets at a price P. The equilibrium asset price P^* satisfies the following zero-profit condition.

$$\Pi(P^*) = \frac{\theta^*(P^*) + \underline{\theta}}{2} - P^* = \frac{1}{2}\left(\frac{D_2 - D_1}{1 - qD_1/P^*} + \underline{\theta}\right) - P^* = 0.$$

[41] The derivation follows the standard global games approach. Details are included in the Online Appendix.

One can verify that $\Pi(P)$ monotonically decreases in P, $\Pi(P) > 0$ when $P = \underline{P}$ and $\Pi(P) < 0$ when $P = qD_2$. Therefore, there exists a unique $P^* \in [\underline{P}, qD_2)$ such that $\Pi(P^*) = 0$. With the unique $P^* \in [\underline{P}, qD_2)$ established, it is then straightforward to verify that the associated $\theta^* \in [\theta^L, \theta^U]$ and $x^* \in [x^L, x^U]$ exist and are unique. In addition, we can derive the following closed-form solutions:

$$\theta^* = \frac{\Psi + \sqrt{\Psi - 8qD_1\underline{\theta}} - 2\underline{\theta}}{2}, \quad x^* = \theta^* + (2q-1)\epsilon, \quad P^* = \frac{\Psi + \sqrt{\Psi - 8qD_1\underline{\theta}}}{4},$$

with $\Psi = (D_2 - D_1) + 2qD_1 + \underline{\theta}$.[42] □

Appendix B.3 Proof of Lemma 3

Proof. The proof consists of two steps.

Step 1: We prove the existence and uniqueness of the combination $(\theta^*(y^{-i}), x^*(y^{-i}), P^*_M)$ as a solution to the system of equations (17) for each $M \in \{1, 2\}$.

For each $M \in \{1, 2\}$, the critical cash flow $\theta^*(y^{-i}) = \theta^*_M$ and the critical signal $x^*(y^{-i}) = x^*_M = \theta^*_M + (2q-1)\epsilon$, if exist, are constants and do not depend on y^{-i}. We let the price P^*_M be the argument and express θ^*_M as follows.

$$\theta^*_M = \frac{D_2 - D_1}{1 - qD_1/P^*_M} \equiv \theta^*(P^*_M), \tag{B.37}$$

We define a function $\Pi_M(P)$ as the asset buyers' expected profit function from purchasing banks' assets at a price P when observing M bank runs.

$$\Pi_M(P) = \omega^B_M(\theta^*(P)) \cdot \left(\frac{\underline{\theta}_B + \theta^*(P)}{2} - P\right) + \omega^G_M(\theta^*(P)) \cdot \left(\frac{\underline{\theta}_G + \theta^*(P)}{2} - P\right).$$

Denote $\pi(P|s) = [\underline{\theta}_s + \theta^*(P)]/2 - P$ as the buyers' unit payoff from purchasing banks' assets in a given state $s \in \{G, B\}$. The equilibrium asset price P^*_M, if exists, must satisfy the asset buyers' break-even condition: $\Pi_M(P^*_M) = \omega^B_M\left(\theta^*(P^*_M)\right) \cdot \pi(P^*_M|B) + \omega^G_M\left(\theta^*(P^*_M)\right) \cdot \pi(P^*_M|G) = 0$.

Since $\omega^B_M(\theta^*(P)) + \omega^G_M(\theta^*(P)) = 1$ for $P \in [\underline{P}, qD_2)$, we can further express $\Pi_M(P)$ as follows:

$$\Pi_M(P) = \frac{\theta^*(P) + \underline{\theta}_G}{2} - \frac{\underline{\theta}_G - \underline{\theta}_B}{2}\omega^B_M(\theta^*(P)) - P.$$

[42] The detailed derivation of (θ^*, x^*, P^*) and the verification can be found in the Online Appendix.

Take the first order derivative with respect to P, we obtain

$$\frac{d\Pi_M(P)}{dP} = \frac{1}{2}\frac{d\theta^*(P)}{dP} - \frac{\underline{\theta}_G - \underline{\theta}_B}{2}\frac{d\omega_M^B(\theta^*(P))}{dP} - 1. \tag{B.38}$$

It is straightforward to check that $d\theta^*(P)/dP < 0$. Moreover, we have

$$\frac{d\omega_M^B(\theta^*(P))}{dP} = -\frac{d\theta^*(P)}{dP} \cdot \frac{M \cdot \kappa \cdot \left(\theta^*(P) - \underline{\theta}_G\right)^{M-1} \cdot \left(\theta^*(P) - \underline{\theta}_B\right)^{M-1} \cdot \left(\underline{\theta}_G - \underline{\theta}_B\right)}{\left[\left(\theta^*(P) - \underline{\theta}_B\right)^M + \kappa\left(\theta^*(P) - \underline{\theta}_G\right)^M\right]^2} > 0, \quad \forall M \in \{1, 2\}.$$

As a consequence, $\Pi_M(P)$ monotonically decreases in P. Since the posterior beliefs $\omega_M^B(\theta^*(P))$ and $\omega_M^G(\theta^*(P))$ are positive and smaller than one for $P \in [\underline{P}, qD_2)$, it can be verified that $\Pi_M(P) > 0$ when $P = \underline{P}$ and $\Pi_M(P) < 0$ when $P = qD_2$. Therefore, there exists a unique $P_M^* \in [\underline{P}, qD_2)$ such that $\Pi_M(P_M^*) = 0$ for each $M \in \{1, 2\}$. With the unique $P_M^* \in [\underline{P}, qD_2)$ established, it is then straightforward to verify that the associated $\theta_M^* \in [\theta^L, \theta^U]$ and $x_M^* \in [x^L, x^U]$ exist and are unique.

Step 2: We prove $\theta_2^* > \theta_1^*$, $x_2^* > x_1^*$ and $P_2^* < P_1^*$.

For simplicity reason, we let the critical cash flow θ be the argument. Denote $P^*(\theta) = qD_1\theta / [\theta - (D_2 - D_1)]$ as the inverse of $\theta^*(P) = (D_2 - D_1)/(1 - qD_1/P)$. We can reformulate the asset buyers' zero-profit condition when observing M bank runs as follows.

$$\Pi_M(\theta_M^*) = \omega_M^B(\theta_M^*) \cdot \pi(\theta_M^*|B) + \omega_2^G(\theta_M^*) \cdot \pi(\theta_M^*|G) = 0. \tag{B.39}$$

Here, $\pi(\theta|s) = \left(\underline{\theta}_s + \theta\right)/2 - P^*(\theta)$ is denoted as the buyers' unit payoff from purchasing banks' assets in a given state $s \in \{G, B\}$. The proof hinges on the monotonicity of two ratios.

$$\frac{\omega_M^B(\theta)}{\omega_M^G(\theta)} = \frac{(\theta - \underline{\theta}_B)^M}{\kappa(\theta - \underline{\theta}_G)^M} \qquad \text{and} \qquad \frac{\pi(\theta|G)}{\pi(\theta|B)} = \frac{(\underline{\theta}_G + \theta)/2 - P^*(\theta)}{(\underline{\theta}_B + \theta)/2 - P^*(\theta)},$$

where $\theta \in [\theta^L, \theta^U]$. The former one is a ratio of posterior beliefs about state when observing M runs, and the latter one is a ratio of buyers' profit across two states. As the inverse function of $\theta^*(P)$, it is obvious that $\partial P^*(\theta)/\partial\theta < 0$. Both ratios strictly decrease in θ when $\theta > D_2 > \underline{\theta}_s$ as

$$\frac{d}{d\theta}\left(\frac{\omega_M^B(\theta)}{\omega_M^G(\theta)}\right) = -\frac{1}{\kappa} \cdot \frac{M \cdot (\theta - \underline{\theta}_B)^{M-1} \cdot (\underline{\theta}_G - \underline{\theta}_B)}{\left(\theta - \underline{\theta}_G\right)^{M+1}} < 0$$

$$\frac{d}{d\theta}\left(\frac{\pi(\theta|G)}{\pi(\theta|B)}\right) = -\frac{[1/2 - \partial P^*(\theta)/\partial\theta]\,(\underline{\theta}_G - \underline{\theta}_B)}{2\left[\frac{\theta + \underline{\theta}_B}{2} - P^*(\theta)\right]^2} < 0$$

Notice that $(\theta - \underline{\theta}_B)/(\theta - \underline{\theta}_G) > 1$ when $\theta > D_2$. Therefore, we have

$$\kappa \cdot \frac{\omega_1^B(\theta)}{\omega_1^G(\theta)} = \left(\frac{\theta - \underline{\theta}_B}{\theta - \underline{\theta}_G}\right) < \left(\frac{\theta - \underline{\theta}_B}{\theta - \underline{\theta}_G}\right)^2 = \kappa \cdot \frac{\omega_2^B(\theta)}{\omega_2^G(\theta)}. \tag{B.40}$$

$\omega_1^B(\theta)/\omega_1^G(\theta) < \omega_2^B(\theta)/\omega_2^G(\theta)$ must hold $\forall \theta > D_2$.

We then prove our result by contradiction. Suppose $\theta_1^* \geq \theta_2^*$. By the monotonicity of $\pi(\theta|G)/\pi(\theta|B)$, we have

$$\frac{\pi(\theta_1^*|G)}{\pi(\theta_1^*|B)} \leq \frac{\pi(\theta_2^*|G)}{\pi(\theta_2^*|B)}. \tag{B.41}$$

By the equilibrium condition (B.39) for $M \in \{1,2\}$, the following equations must be true, respectively.

$$\frac{\pi(\theta_1^*|G)}{\pi(\theta_1^*|B)} = -\frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} \quad \text{and} \quad \frac{\pi(\theta_2^*|G)}{\pi(\theta_2^*|B)} = -\frac{\omega_2^B(\theta_2^*)}{\omega_2^G(\theta_2^*)}.$$

Together with (B.41), we obtain the following inequality.

$$\frac{\omega_2^B(\theta_2^*)}{\omega_2^G(\theta_2^*)} \leq \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)}.$$

By inequality (B.40), we further have

$$\frac{\omega_2^B(\theta_2^*)}{\omega_2^G(\theta_2^*)} \leq \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} < \frac{\omega_2^B(\theta_1^*)}{\omega_2^G(\theta_1^*)}.$$

Lastly, by the monotonicity of $\omega_2^B(\theta)/\omega_2^G(\theta)$, $\theta_2^* > \theta_1^*$, a contraction. Therefore, $\theta_2^* > \theta_1^*$ must hold. It follows directly $P_2^* < P_1^*$ from the monotonicity of $P^*(\theta)$, and $x_2^* = \theta_2^* + (2q-1)\epsilon > \theta_1^* + (2q-1)\epsilon = x_1^*$.

□

Appendix B.4 Proof of Proposition 2

Proof. Having established that $x^*(y^{-i}) = x_1^*$ when $y^{-i} > \hat{y}$ and $x^*(y^{-i}) = x_2^*$ when $y^{-i} < \hat{y}$, where $\hat{y} \in [y^L, y^U]$, we show here there exists a unique $\hat{y} = x_2^* + \eta + \epsilon$. The proof hinges on deriving a bank i's representative creditor j's rational expectations on the fractions of withdrawals (i.e., L^i and L^{-i}) based on his signals (x_j^i, y^{-i}). We analyze whether $L^{-i}(x_j^i, y^{-i})$ is non-zero to determine the creditor's expectations on the number of runs M and the equilibrium asset price.[43] Note $L^{-i}(x_j^i, y^{-i}) = E\left\{E\left[L^{-i}(\theta^{-i}, x^*(y^i))\middle|y^{-i}\right]\middle|x_j^i\right\}$ is given by (A.35) in Appendix A.2.

Step 1: We prove that upon observing $y^{-i} \in [y^L, x_1^* - \eta + \epsilon]$ and $y^{-i} \in [x_2^* + \eta + \epsilon, y^U]$, the representative creditor expects $L^{-i}(x_j^i, y^{-i}) > 0$ and $L^{-i}(x_j^i, y^{-i}) = 0$ respectively, both independent of his private signal x_j^i.

We derive the representative creditor j's expectation of $L^{-i}(x_j^i, y^{-i})$ when he observes $y^{-i} = x_1^* - \eta + \epsilon$ and has a posterior $\theta^{-i}\big|_{y^{-i}} \sim U(x_1^* - 2\eta + \epsilon, x_1^* + \epsilon)$. Note that the creditors in the bank $-i$ can follow either x_2^* or x_1^* as the critical signal depending on their outside signal y^i. If those creditors withdraw according to the critical signal x_1^* (i.e., $x^*(y^i) = x_1^*$), we can compute the expectation $E\left[L^{-i}(\theta^{-i}, x^*(y^i))\middle|y^{-i}\right]$ in the expression of $L^{-i}(x_j^i, y^{-i})$ as follows:[44]

$$E\left[L^{-i}(\theta^{-i}, x_1^*)\middle|x_1^* - \eta + \epsilon\right] = \int_{x_1^*-2\eta+\epsilon}^{x_1^*-\epsilon} 1 \cdot \frac{1}{2\eta} d\theta^{-i} + \int_{x_1^*-\epsilon}^{x_1^*+\epsilon} \frac{x_1^* - \theta^{-i} + \epsilon}{2\epsilon} \cdot \frac{1}{2\eta} d\theta^{-i} = \frac{2\eta - \epsilon}{2\eta} > 0.$$

Instead, if they withdraw according to x_2^*, one can verify easily that $L^{-i}(\theta^{-i}, x_2^*) = 1$ and

$$E\left[L^{-i}(\theta^{-i}, x_2^*)\middle|x_1^* - \eta + \epsilon\right] = \int_{x_1^*-2\eta+\epsilon}^{x_1^*+\epsilon} L^{-i}(\theta^{-i}, x_2^*) \cdot \frac{1}{2\eta} \cdot d\theta^{-i} = 1.$$

Either way, the representative creditor expects $L^{-i}(x_j^i, y^{-i}) > 0$, because $f_{Y^i}(y^i|x_j^i)$ is everywhere non-negative, and it is strictly positive when $y^i \in [x_j^i - \eta - \epsilon, x_j^i + \eta + \epsilon]$. By the monotonicity of $L^{-i}(x_j^i, y^{-i})$ with respect to y^{-i}, we establish that $L^{-i}(x_j^i, y^{-i}) > 0$ when $y^{-i} \in [y^L, x_1^* - \eta + \epsilon]$.

Consider the representative creditor j's expectation of $L^{-i}(x_j^i, y^{-i})$ when observing $y^{-i} > x_2^* + \eta + \epsilon$. He knows with certainty that the lowest possible private signal received by the bank $-i$'s creditors is still higher than x_2^*. The representative creditor then expects $L^{-i}(x_j^i, y^{-i}) = 0$. Follow the same argument in Lemma 3, we can obtain $x^*(y^{-i}) = x_2^*$ when $y^{-i} \in [y^L, x_1^* - \eta + \epsilon]$ and $x^*(y^{-i}) = x_1^*$ when $y^{-i} \in (x_2^* + \eta + \epsilon, y^U]$.

Step 2: Given that all other creditors follow the strategy (18), we prove that the representative creditor j expects $L^{-i}(x_j^i, y^{-i}) > 0$ when observing $y^{-i} \in (x_1^* - \eta + \epsilon, x_2^* + \eta + \epsilon)$ and $x_j^i \leq x_2^* + 2\epsilon$. We prove this result by first establishing that it holds for $x_j^i = x_2^* + 2\epsilon$. Then by the monotonicity of $L^{-i}(x_j^i, y^{-i})$ in x_j^i established in Appendix A.3, we have $L^{-i}(x_j^i, y^{-i}) > 0$, $\forall x_j^i \leq x_2^* + 2\epsilon$.

[43] To determine whether $L^i(x_j^i, y^{-i})$ is non-zero is less involving. In particular, the representative creditor knows that $L^i(x_j^i, y^{-i}) > 0$ when $x_j^i \leq x^*(y^{-i}) + 2\epsilon$ and $L^i(x_j^i, y^{-i}) = 0$ when $x_j^i > x^*(y^{-i}) + 2\epsilon$ from Appendix A.2.

[44] Note that $L^{-i}(\theta^{-i}, x_1^*) = 1$ if $x_1^* - 2\eta + \epsilon < \theta^{-i} < x_1^* - \epsilon$ and $L^{-i}(\theta^{-i}, x_1^*) = (x_1^* - \theta^{-i} + \epsilon)/(2\epsilon)$ if $x_1^* - \epsilon < \theta^{-i} < x_1^* + \epsilon$.

Upon observing $x_j^i = x_2^* + 2\epsilon$, the representative creditor forms beliefs about the outside signal y^i received by the other bank's creditors. He can calculate, by (A.34), that y^i has a positive conditional density $f_{Y^i}(y^i|x_2^* + 2\epsilon)$ on the interval $[x_2^* - \eta + \epsilon, x_2^* + \eta + 3\epsilon]$.

$$f_{Y^i}(y^i|x_2^* + 2\epsilon) = \begin{cases} \frac{y^i - (x_2^* - \eta + \epsilon)}{4\eta\epsilon} & x_2^* - \eta + \epsilon < y^i \le x_2^* - \eta + 3\epsilon \\ \frac{1}{2\eta} & x_2^* - \eta + 3\epsilon < y^i \le x_2^* + \eta + \epsilon \\ \frac{(x_2^* + \eta + 3\epsilon) - y^i}{4\eta\epsilon} & x_2^* + \eta + \epsilon < y^i \le x_2^* + \eta + 3\epsilon \\ 0 & \text{otherwise.} \end{cases}$$

Under strategy (18), the bank $-i$'s creditors will follow the critical signal $x^*(y^i) = x_2^*$ when they observe $y^i < x_2^* + \eta + \epsilon$ and $x^*(y^i) = x_1^*$ when they observe $y^i \ge x_2^* + \eta + \epsilon$. In particular, the representative creditor expects that the creditors in the bank $-i$ will follow x_2^* with a probability

$$Prob\left(x^*(y^i) = x_2^*|x_j^i = x_2^* + 2\epsilon\right) = \int_{x_2^*-\eta+\epsilon}^{x_2^*-\eta+3\epsilon} \frac{y^i - (x_2^* - \eta + \epsilon)}{4\eta\epsilon} dy^i + \int_{x_2^*-\eta+3\epsilon}^{x_2^*+\eta+\epsilon} \frac{1}{2\eta} dy^i = \frac{2\eta - \epsilon}{2\eta} > 0.$$

Conditional on that creditors from the bank $-i$ actually follow the critical signal x_2^*, the representative creditor can calculate the expected aggregate withdrawals in the bank $-i$ as $E[L^{-i}(\theta^{-i}, x_2^*)|y^{-i}] = \int_{y^{-i}-\eta}^{y^{-i}+\eta} L^{-i}(\theta^{-i}, x_2^*) \cdot \frac{1}{2\eta} \cdot d\theta^{-i}$ since $\theta^{-i}|_{y^{-i}} \sim U(y^{-i} - \eta, y^{-i} + \eta)$. When the representative creditor observes an outside signal $x_1^* - \eta + \epsilon < y^{-i} < x_2^* + \eta + \epsilon$, he knows that the expected aggregate withdrawal has a lower bound

$$E\left[L^{-i}(\theta^{-i}, x_2^*)\middle|y^{-i}\right] \ge \int_{y^{-i}-\eta}^{x_2^*+\epsilon} \frac{x_2^* - \theta^{-i} + \epsilon}{2\epsilon} \cdot \frac{1}{2\eta} \cdot d\theta^{-i} = \frac{(x_2^* + \eta + \epsilon - y^{-i})^2}{8\eta\epsilon} > 0. \tag{B.42}$$

The first inequality in (B.42) is true because $E\left[L^{-i}(\theta^{-i}, x_2^*)\middle|y^{-i}\right]$ decreases in y^{-i} and the functional form of $L^{-i}(\theta^{-i}, x_2^*)$ is $(x_2^* - \theta^{-i} + \epsilon)/(2\epsilon)$ if $y^{-i} - \eta \le \theta^{-i} < x_2^* + \epsilon$ and 0 if $x_2^* + \epsilon \le \theta^{-i} \le y^{-i} + \eta$, conditional on y^{-i} slightly lower than $x_2^* + \eta + \epsilon$ (i.e., $y^{-i} \in (x_2^* + \eta - \epsilon, x_2^* + \eta + \epsilon)$).

When the fundamentals (θ^1, θ^2) are realized, $\theta^{-i}|_{y^{-i}}$ and $y^i|_{x_j^i}$ are independent. From (A.35), a representative creditor observing $x_j^i = x_2^* + 2\epsilon$ and $y^{-i} \in (x_1^* - \eta + \epsilon, x_2^* + \eta + \epsilon)$, will calculate

$$\begin{aligned} L^{-i}(x_2^* + 2\epsilon, y^{-i}) &= \int_{x_2^*-\eta+\epsilon}^{x_2^*+\eta+\epsilon} E\left[L^{-i}(\theta^{-i}, x_2^*)\middle|y^{-i}\right] \cdot f_{Y^i}(y^i|x_2^* + 2\epsilon) \cdot dy^i + \int_{x_2^*+\eta+\epsilon}^{x_2^*+\eta+3\epsilon} E\left[L^{-i}(\theta^{-i}, x_1^*)\middle|y^{-i}\right] \cdot f_{Y^i}(y^i|x_2^* + 2\epsilon) \cdot dy^i \\ &\ge \int_{x_2^*-\eta+\epsilon}^{x_2^*+\eta+\epsilon} \frac{(x_2^* + \eta + \epsilon - y^{-i})^2}{8\eta\epsilon} \cdot f_{Y^i}(y^i|x_2^* + 2\epsilon) \cdot dy^i \\ &= \frac{(x_2^* + \eta + \epsilon - y^{-i})^2}{8\eta\epsilon} \cdot Prob\left(x^*(y^i) = x_2^*|x_j^i = x_2^* + 2\epsilon\right) \\ &= \frac{(x_2^* + \eta + \epsilon - y^{-i})^2}{8\eta\epsilon} \cdot \frac{2\eta - \epsilon}{2\eta} > 0. \end{aligned}$$

The first inequality follows (B.42), the fact that $E\left[L^{-i}(\theta^{-i}, x_1^*)\middle|y^{-i}\right] \geq 0$[45], and the density of y^i being non-negative everywhere.

Step 3: We establish the existence of the equilibrium strategy (18) by analyzing the representative creditor's best response.

We have already proved that $x^*(y^{-i}) = x_2^*$ for $y^{-i} \leq x_1^* - \eta + \epsilon$ and $x^*(y^{-i}) = x_1^*$ for $y^{-i} \geq x_2^* + \eta + \epsilon$. Given that all other creditors follow the strategy (18), the representative creditor expects $L^i(x_j^i, y^{-i}) > 0$ when $x_j^i \leq x_2^* + 2\epsilon$, i.e., a positive mass of withdrawals in his own bank. For $y^{-i} \in (x_1^* - \eta + \epsilon, x_2^* + \eta + \epsilon)$ and $x_j^i \leq x_2^* + 2\epsilon$, the creditor expects the number of runs to be $M = 2$ and the asset price to be P_2^*. Instead, when observing $x_j^i > x_2^* + 2\epsilon$, he expects $L^i(x_j^i, y^{-i}) = 0$, i.e., no run in his own bank. We can follow the same procedure in the proof of Proposition 1 and of Lemma 3, to show that the representative creditor optimally withdraws if and only if $x_j^i < \hat{x}(y^{-i})$, where $\hat{x}(y^{-i}) = x_2^* - 2\epsilon[L^c(\theta^*(y^{-i}), P_2^*) - q]$ for $\forall y^{-i} \in [y^L, x_2^* + \eta + \epsilon)$. A symmetric equilibrium, if exists, features $L^c(\theta^*(y^{-i}), P_2^*) = q$. The bank's critical cash flow, the bank i's creditors' threshold signal, and the asset price, jointly solve the system of equations (17) for $M = 2$, with the unique solution being $\left(\theta_2^*, x_2^*, P_2^*\right)$ by Lemma 3. To summarize, when all other creditors in the bank i and $-i$ take the strategy (18), the representative creditor's best response is to follow the same strategy.

Step 4: We prove the uniqueness by contradiction. Suppose that all other creditors follow a threshold strategy with a discontinuity point $\hat{y} < x_2^* + \eta + \epsilon$, so that $x^*(y^{-i}) = x_1^*$ when $y^{-i} \in (\hat{y}, x_2^* + \eta + \epsilon)$.

Suppose that the representative creditor j observes a private signal $x_j^i = x_1^*$. Following the same procedure in Step 2, the probability that creditors in the bank $-i$ following the critical signal x_2^* is positive: $Prob\left(x^*(y^i) = x_2^* | x_j^i = x_1^*\right) = \int_{x_1^*-\eta-\epsilon}^{x_1^*-\eta+\epsilon} \frac{y^i-(x_1^*-\eta-\epsilon)}{4\eta\epsilon} \cdot dy^i = \frac{\epsilon}{2\eta} > 0$. Conditional on that creditors from the bank $-i$ follow the threshold signal x_2^* and that the representative creditor observes $y^{-i} \in (\hat{y}, x_2^* + \eta + \epsilon)$, the aggregate withdrawals again satisfy the inequality (B.42). The representative creditor j rationally expects $L^{-i}(x_1^*, y^{-i})$ to be strictly positive.

$$L^{-i}(x_1^*, y^{-i}) \geq \frac{(x_2^* + \eta + \epsilon - y^{-i})^2}{8\eta\epsilon} \cdot \frac{\epsilon}{2\eta} > 0.$$

By monotonicity, the representative creditor expects $M = 2$ and the secondary market asset price to be P_2^* when observing $x_j^i \leq x_1^*$ and $y^{-i} \in (\hat{y}, x_2^* + \eta + \epsilon)$. Consequently, the representative

[45] The representative creditor also expects that the creditors in the bank $-i$ will follow x_1^* with a probability $\int_{x_2^*+\eta+\epsilon}^{x_2^*+\eta+3\epsilon} f_{Y^i}(y^i | x_2^* + 2\epsilon) \cdot dy^i$ and calculates the expected aggregate withdrawals in the bank $-i$ as $E\left[L^{-i}(\theta^{-i}, x_1^*)\middle|y^{-i}\right]$.

creditor expects his own bank (i.e., the bank i) to sell its assets for the price P_2^* in case runs happen. The expected price P_2^*, however, contradicts the threshold signal x_1^* as dictated by the alternative threshold strategy. Therefore, we establish the uniqueness by contradiction.

□

Appendix B.5 Proof of Proposition 4

Proof. Note that one can reformulate $V_A(P_A)$ in (24) as follows.

$$V_A(P_A) = \sum_{s=G,B} Pr(s) \cdot 2 \cdot \left(Pr(\theta < \theta^*(P_A)) \cdot \frac{D_1}{P_A} \cdot \pi(P_A|s) \right). \tag{B.43}$$

We then take the first order derivative with respect to P_A.

$$\frac{dV_A(P_A)}{dP_A} = \sum_{s=G,B} \frac{Pr(s)}{\bar{\theta} - \underline{\theta}_s} \cdot 2 \cdot \left\{ \frac{d}{dP_A} \left[\left(\theta^*(P_A) - \underline{\theta}_s \right) \cdot \pi(P_A|s) \right] \cdot \frac{D_1}{P_A} - \frac{D_1}{(P_A)^2} \left(\theta^*(P_A) - \underline{\theta}_s \right) \cdot \pi(P_A|s) \right\}.$$

Insert $\pi(P_A|s) = (\underline{\theta}_s + \theta^*(P_A))/2 - P_A$, we obtain

$$\frac{dV_A(P_A)}{dP_A} = \sum_{s=G,B} \frac{Pr(s)}{\bar{\theta} - \underline{\theta}_s} \cdot 2 \cdot \left\{ \frac{d\theta^*(P_A)}{dP_A} \cdot (\theta^*(P_A) - P_A) - \frac{\left(\theta^*(P_A) + \underline{\theta}_s \right)\left(\theta^*(P_A) - \underline{\theta}_s \right)}{2P_A} \right\} \cdot \frac{D_1}{P_A} < 0.$$

Note that $\theta^*(P_A) > D_2 > P_A > \underline{\theta}_s$ when $P_A \in [\underline{P}, qD_2)$ and $d\theta^*(P_A)/dP_A = -[qD_1(D_2 - D_1)]/(P_A - qD_1)^2 < 0$. We have established that there exists a $P_A^* \in (P_2^*, P_1^*)$ such that $V_A(P_A^*) = 0$. By the monotonicity of $V_A(P_A)$, such P_A^* is also unique.

From the expression of $\mathrm{SYS}(P_A)$ in (23), we have

$$\frac{d\mathrm{SYS}(P_A)}{dP_A} = \sum_{s=G,B} \frac{Pr(s)}{(\bar{\theta} - \underline{\theta}_s)^2} \cdot 2 \cdot \left(\theta^*(P_A) - \underline{\theta}_s \right) \cdot \frac{d\theta^*(P_A)}{dP_A} < 0.$$

The regulator's objective $\mathrm{SYS}(P_A)$ monotonically decreases in P_A, so she optimally chooses P_A^* such that the constraint (20) is binding, i.e., $V_A(P_A^*) = 0$. It is easy to see that the constraint (21) is slack since the constraint requires $P_A \leq qD_2$. And we have $\mathrm{SYS}(P_A^*) < \mathrm{SYS}(\theta_2^*)$ as $P_A^* > P_2^*$.

□

Appendix B.6 Proof of Proposition 5

Proof. We denote $a_i \in \{A, NA\}$, $i \in \{1, 2\}$, as a bank i's manager's action, where 'A' and 'NA' stand for 'accept the regulator's offer' and 'refuse the regulator's offer' respectively. At the start of $t = 1$, the regulator's offer $P_A \in (P_2^*, P_1^*)$ and two bank managers' strategy profile (a_i, a_{-i}) define a subgame $g_{a_i,a_{-i}}(P_A)$. We establish a condition under which both banks accepting the contract for the regulator's liquidity support is a Nash equilibrium.

Step 1: We derive the manager i's equilibrium payoffs in subgame $g_{A,A}(P_A)$ and $g_{NA,A}(P_A)$. In subgame $g_{A,A}(P_A)$, the analyses are exactly the same as that in Section 3.3.2. The equilibrium threshold signal $x^*(P_A)$ and critical cash flow $\theta^*(P_A)$ are given by (22).

In subgame $g_{NA,A}(P_A)$, only the bank $-i$'s manager signs the contract for the regulator's liquidity support. Knowing with certainty that the bank's asset price will be P_A, creditors in the bank $-i$ no longer need to form rational beliefs about the number of runs (i.e., the outside signal y^i becomes redundant). The equilibrium threshold signal and critical cash flow for the bank $-i$ will be $x^*(P_A)$ and $\theta^*(P_A)$.

The bank i's manager does not sign the contract with the regulator. Upon runs happen, the bank still raises liquidity from the secondary market where the asset buyers bid competitively to purchase the bank's assets on sale. The analyses of the bank run game and the asset bidding game resemble that in Section 3.2.2, the bank i's critical cash flow $\theta^*(y^{-i})$, the creditors' threshold signal $x^*(y^{-i})$, and the equilibrium asset price $P_{A,M}^*$ jointly solve the following system of equations.

$$\begin{cases} P_{A,M}^* = \omega_M^B(\theta_{A,M}^*, P_A) \cdot \frac{\underline{\theta}_B + \theta_{A,M}^*}{2} + \omega_M^G(\theta_{A,M}^*, P_A) \cdot \frac{\underline{\theta}_G + \theta_{A,M}^*}{2} \\ x^*(y^{-i}) = \theta^*(y^{-i}) + (2q-1)\epsilon \\ \theta^*(y^{-i}) = \frac{D_2 - D_1}{1 - qD_1/P_{A,M}^*} \\ \theta^*(y^{-i}) = \theta_{A,M}^* \end{cases} \quad \text{with} \quad M = \begin{cases} 1 \\ 2 \end{cases} \tag{B.44}$$

When observing M runs, the asset buyers believe that the bank i's cash flow is lower than a threshold cash flow $\theta_{A,M}^*$. They also Bayesian update their beliefs about the aggregate state s. Their posteriors beliefs about s are characterized by (B.45).

$$\omega_M^B(\theta_{A,M}^*, P_A) = \frac{\left(\frac{\theta_{A,M}^* - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right) \cdot \omega_{M-1}^B(P_A)}{\left(\frac{\theta_{A,M}^* - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right) \cdot \omega_{M-1}^B(P_A) + \left(\frac{\theta_{A,M}^* - \underline{\theta}_G}{\bar{\theta} - \underline{\theta}_G}\right) \cdot \omega_{M-1}^G(P_A)} = 1 - \omega_M^G(\theta_{A,M}^*, P_A), \tag{B.45}$$

where

$$\omega_1^B(P_A) = Prob(s = B | \theta^{-i} < \theta^*(P_A)) = \frac{\left(\frac{\theta^*(P_A) - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right) \cdot (1-\alpha)}{\left(\frac{\theta^*(P_A) - \underline{\theta}_B}{\bar{\theta} - \underline{\theta}_B}\right) \cdot (1-\alpha) + \left(\frac{\theta^*(P_A) - \underline{\theta}_G}{\bar{\theta} - \underline{\theta}_G}\right) \cdot \alpha} = 1 - \omega_1^G(P_A),$$

$$\omega_0^B(P_A) = Prob(s = B | \theta^{-i} > \theta^*(P_A)) = \frac{\left(\frac{\bar{\theta} - \theta^*(P_A)}{\bar{\theta} - \underline{\theta}_B}\right) \cdot (1-\alpha)}{\left(\frac{\bar{\theta} - \theta^*(P_A)}{\bar{\theta} - \underline{\theta}_B}\right) \cdot (1-\alpha) + \left(\frac{\bar{\theta} - \theta^*(P_A)}{\bar{\theta} - \underline{\theta}_G}\right) \cdot \alpha} = 1 - \omega_0^G(P_A).$$

Conditional on whether a run happens to the bank $-i$, the buyers' beliefs about s become $\omega_1^s(P_A)$ or $\omega_0^s(P_A)$. When being called upon to purchase the bank i's assets, the buyers update beliefs once more. Their posteriors about s become $\omega_2^s(\theta_{A,2}^*, P_A)$ when runs happen to both banks or $\omega_1^s(\theta_{A,1}^*, P_A)$ when a run only happens to the bank i. We have the following Result 2.[46]

Result 2. *When $P_A \in (P_2^*, P_1^*)$, the combination $\left(\theta_{A,M}^*, x_{A,M}^*, P_{A,M}^*\right)$ is the unique solution to the system of equations (B.44) for each $M \in \{1,2\}$. It holds that $P_{A,2}^* < P_2^*$, $\theta_{A,2}^* > \theta_2^*$ and $x_{A,2}^* > x_2^*$, while $P_{A,1}^* = P_1^*$, $\theta_{A,1}^* = \theta_1^*$ and $x_{A,1}^* = x_1^*$. A bank who signs the contract for the regulator's liquidity support exerts only a negative externality on the other bank who does not sign the contract.*

Result 2 compares the solution of the system (B.44) with that of the system (17) . In particular, we have $P_{A,2}^* < P_2^*$ and $\theta_{A,2}^* > \theta_2^*$. Intuitively, in case of two bank runs, the demise of the bank who accepts the regulator's price support conveys more adverse information about s. Indeed, the asset buyers' posteriors about s deteriorate when observing a run also happens to the bank $-i$, i.e., $\omega_2^B(\theta, P_A) > \omega_2^s(\theta)$, $\forall\theta \in [\theta^L, \theta^U]$. Instead, we have $P_{A,1}^* = P_1^*$ and $\theta_{A,1}^* = \theta_1^*$. The survival of a bank (i.e., its cash flow is beyond a threshold cash flow) conveys no information about the (downside) aggregate risk factor, that is, whether $\underline{\theta}_s$ equals $\underline{\theta}_G$ or $\underline{\theta}_B$. The asset buyers' posteriors about s remain the same when observing the survival of the bank $-i$, i.e., $\omega_1^s(\theta, P_A) = \omega_1^s(\theta)$, $\forall\theta \in [\theta^L, \theta^U]$.

In Result 3, we characterize the bank $-i$'s creditors' equilibrium threshold strategy and the asset buyers' equilibrium bids for the bank's assets on sale.

Result 3. *When the bank i does not accept the regulator's offer $P_A \in (P_2^*, P_1^*)$, its bank run game has a unique equilibrium. A representative creditor of the bank withdraws early if and only if his private signal falls below*

$$x^*(y^{-i}, P_A) = \begin{cases} x_{A,2}^* & y^{-i} < x^*(P_A) + \eta + \epsilon \\ x_1^* & y^{-i} \geq x^*(P_A) + \eta + \epsilon. \end{cases}$$

The asset buyers offer price $P_{A,2}^*$ to purchase the bank i's assets when observing two bank runs and offer price P_1^* when observing only one run happening to the bank i.

The result has a similar intuition as that of Proposition 2, the creditors in the bank i rely on the outside signal y^{-i} to form a rational expectation on the equilibrium asset price. In the

[46]The proofs of Result 2 and Result 3 mirror the ones of Lemma 3 and Proposition 2. We provide the details in the Online Appendix to save the space.

limiting case where η and ϵ approach zero, we have $x^*_{A,M} = \theta^*_{A,M} = (D_2 - D_1)/(1 - qD_1/P^*_{A,M})$ and $x^*(P_A) = \theta^*(P_A) = (D_2 - D_1)/(1 - qD_1/P_A)$.

Step 2: We analyze the bank i's manager's optimal choice regarding whether to accept the regulator's offer P_A.

We move back to $t = 0$ to consider the optimal response of the bank i's manager (thereafter the manager i) to the bank $-i$'s manager's action $a_{-i} = A$. The manager i's objective is $[1 - Pr(\text{failure of the bank i})] \times b$, where

$$Pr(\text{failure of bank i}|a_{-i} = A) = \begin{cases} Pr(A, A) & \text{if } a_i = A, \\ Pr(NA, A) & \text{if } a_i = NA. \end{cases}$$

If $Pr(A, A) < Pr(NA, A)$, the manager i's best response is $a^*_i = A$, the strategy profile $(a^*_i, a^*_{-i}) = (A, A)$ becomes a Nash equilibrium of the stage game.

We derive explicitly $Pr(A, A)$ and $Pr(NA, A)$. It can be seen easily

$$Pr(A, A) = \sum_{s=G,B} Pr(s) \cdot \left(\frac{\theta^*(P_A) - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right).$$

Consider the manager i's choice $a_i = NA$. The bank i fails if and only if $\theta^i < \theta^*_{A,2}$ conditional on $\theta^{-i} < \theta^*(P_A)$, while it fails if and only if $\theta^i < \theta^*_{A,1}$ conditional on $\theta^{-i} \geq \theta^*(P_A)$. The bank i's probability of failure is

$$Pr(NA, A) = \sum_{s=G,B} Pr(s) \cdot \left[\left(\frac{\theta^*_{A,1} - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) \cdot \left(\frac{\bar{\theta} - \theta^*(P_A)}{\bar{\theta} - \underline{\theta}_s} \right) + \left(\frac{\theta^*_{A,2} - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) \cdot \left(\frac{\theta^*(P_A) - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) \right].$$

Define an auxiliary function $F(P_A) = Pr(A, A)(P_A) - Pr(NA, A)(P_A)$. We have

$$F(P_A) = \sum_{s=G,B} Pr(s) \cdot \left[\left(\frac{\theta^*(P_A) - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) - \left(\frac{\theta^*_{A,1} - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) \cdot \left(\frac{\bar{\theta} - \theta^*(P_A)}{\bar{\theta} - \underline{\theta}_s} \right) - \left(\frac{\theta^*_{A,2} - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) \cdot \left(\frac{\theta^*(P_A) - \underline{\theta}_s}{\bar{\theta} - \underline{\theta}_s} \right) \right].$$

We establish a $P^C_A \in (P^*_2, P^*_1)$ such that $F(P^C_A) = 0$. When the regulator's price support $P_A = P^C_A$, the manager i is indifferent between his choices A and NA, given $a_{-i} = A$. We have

$$\lim_{P_A \to P^*_2} F(P_A) = \sum_{s=G,B} \frac{Pr(s)}{(\bar{\theta} - \underline{\theta}_s)^2} \cdot (\theta^*_2 - \theta^*_1) \cdot (\bar{\theta} - \theta^*_2) > 0,$$
$$\lim_{P_A \to P^*_1} F(P_A) = \sum_{s=G,B} \frac{Pr(s)}{(\bar{\theta} - \underline{\theta}_s)^2} \cdot (\theta^*_1 - \underline{\theta}_s) \cdot (\theta^*_1 - \lim_{P_A \to P^*_1} \theta^*_{A,2}) < 0.$$

Note that $\theta^*_{A,2} > \theta^*_2$ for $P_A < P^*_2$, so $\lim_{P_A \to P^*_1} \theta^*_{A,2} > \theta^*_2 > \theta^*_1$ must be true. Moreover, we have $\lim_{P_A \to P^*_2} \theta^*_{A,2} = \theta^*_2$ because the secondary market asset buyers' posteriors about s

$\omega_2^s(\theta, \theta^*(P_A)) \to \omega_2^s(\theta, \theta_2^*)$ when $P_A \to P_2^*$. So the equilibrium critical cash flow $\theta_{A,2}^*$ converges to θ_2^* when $P_A \to P_2^*$. By the continuity of $F(P_A)$, there must exist a $P_A^C \in (P_2^*, P_1^*)$ such that $F(P_A^C) = 0$.

On the other hand, it can be checked that

$$\frac{dF(P_A)}{dP_A} = \sum_{s=G,B} \frac{Pr(s)}{(\bar{\theta} - \underline{\theta}_s)^2} \cdot \left[\frac{d\theta^*(P_A)}{dP_A} \cdot \left((\bar{\theta} - \theta_2^*(P_A)) + (\theta_1^* - \underline{\theta}_s) \right) - \frac{d\theta_{A,2}^*}{dP_A} \cdot \left(\theta^*(P_A) - \underline{\theta}_s \right) \right] < 0.$$

We have $d\theta_{A,2}^*/dP_A > 0$ because $\omega_2^B(\theta, \theta^*(P_A))$ increases in P_A. Conditional on a run happening to the bank $-i$, the asset buyers' posterior beliefs about s deteriorate as the regulator's price P_A becomes higher. As a consequence, P_A^C is also unique.

From the above analyses, $F(P_A) < 0$ when $P_A \in (P_A^C, P_1^*)$, and $F(P_A) \geq 0$ when $P_A \in (P_2^*, P_A^C]$. To minimize the bank i's probability of failure, the manager i optimally chooses $a_i^* = A$ when $P_A \in (P_A^C, P_1^*)$ and $a_i^* = NA$ when $P_A \in (P_2^*, P_A^C]$.

Lastly, under a sufficient condition $V_A(P_A^C) > 0$, the regulator's optimal price support P_A^* can induce both banks' managers to chooses the action A. To see this, recall that $V_A'(P_A) < 0$ and the optimal P_A^* satisfying the regulator's break-even condition $V_A(P_A^*) = 0$. Therefore, we have $P_A^* > P_A^C$, in which case $F(P_A^*) < 0$ and $Pr(A, A) < Pr(NA, A)$. Both banks' managers accept the liquidity support offered by the regulator becomes a Nash equilibrium of the game.

□

Appendix B.7 Proof Corollary 2

Proof. We solve the regulator's program when she can take expected losses, i.e., her loss-bearing capacity $\overline{V} > 0$.

Since both $\text{SYS}(P_A)$ and $V_A(P_A)$ monotonically decrease in P_A when $P_A \in [\underline{P}, qD_2)$, the regulator chooses the highest possible P_A to minimize the risk of systemic bank failures $\text{SYS}(P_A)$. Now, the constraint (20) is replaced by $V_A(P_A) \geq -\overline{V}$ and the constraint (21) implies $P_A \leq qD_2$. The optimal price support depends on which of the two constraints will be binding.

Define a cutoff $\overline{V}^C$ such that $V_A(qD_2) = -\overline{V}^C$. Since $V_A(qD_2) < 0$, $\overline{V}^C$ is positive. It is obvious that when $\overline{V} < \overline{V}^C$, $V_A(P_A) \geq -\overline{V}$ is the binding constraint. In this case, the regulator optimally chooses a price $P_A^{**} \in (P_A^*, qD_2)$ such that $V_A(P_A^{**}) = -\overline{V}$. The existence and the uniqueness of P_A^{**} is again guaranteed by the continuity and the monotonicity of $V_A(P_A)$. When $\overline{V} \geq \overline{V}^C$, $P_A \leq qD_2$ will be the binding constraint. The regulator optimally chooses the price

qD_2 even if her loss-bearing capacity has not been exhausted, i.e., $V_A(qD_2) = -\overline{V}^C \geq -\overline{V}$. This is because the regulator would like to avoid saving insolvent banks. □

Appendix B.8 Proof of Lemma 4

Proof. We solve the regulator's program when she unilaterally commits to purchasing banks' assets on sale at $t = 1$ at a price P_U.

We characterize the regulator's program. The regulator has the same objective under both arrangements. Directly following Appendix B.5, $\text{SYS}(P_U)$ is continuous and monotonically decreasing in P_U for $P_U \geq P_2^*$. We then show that $V_U(P_U)$ is also continuous and monotonically decreasing in P_U. From (27), we have:

$$V_U(P_U) = V_A(P_U) = \sum_{M=1}^{2}\left(\sum_{s=G,B} Pr(s)\cdot Pr(\theta < \theta^*(P_U)|s)^M \cdot Pr(\theta > \theta^*(P_U)|s)^{2-M}\right)C_2^M \cdot M \cdot \frac{D_1}{P_U}\cdot \Pi_M(P_U), \quad \text{(B.46)}$$

for $P_U \geq P_1^*$. On the other hand, for $P_U \in [P_2^*, P_1^*)$, $V_U(P_U)$ is given by (26), which can be re-arranged into

$$V_U(P_U) = \left(\sum_{s=G,B} Pr(s)\cdot Pr(\theta < \theta^*(P_U)|s)^2 \cdot 2 \cdot \frac{D_1}{P_U}\right)\cdot \Pi_2(P_U). \quad \text{(B.47)}$$

When $P_U = P_1^*$, (B.46) turns into (B.47) since $\Pi_1(P_1^*) = 0$. Note that $V_U(P)$ has the same functional form as $V_A(P)$ when $P > P_1^*$, so it monotonically decreases in P_U when $P_U > P_1^*$. When $P_U \in [P_2^*, P_1^*)$, one can verify that $V_U(P_U)$ (given by (26)) is also monotonically decreasing in P_U provided that α, the probability of $s = G$, is sufficiently large. We suppose this is true throughout the paper.

To solve the regulator's optimal price under the arrangement with her unilateral commitment, consider first the case $\overline{V} = 0$. It is never optimal for the regulator to pre-commit to a price $P_U \geq P_1^*$ since her expected payoff (given by (27)) will be negative. Instead, by pre-committing to a price $P_U \in [P_2^*, P_1^*)$, the regulator's expected payoff $V_U(P_U)$ is given by (B.47). By monotonicity, the only price that allows the regulator to break even in expectation is $P_U^* = P_2^*$. Then, consider the case $\overline{V} > 0$. Having established the continuity and monotonicity, the solution of the program resembles that in Appendix B.7. When $0 < \overline{V} < -V_U(qD_2)$, the regulator's program has a unique solution $P_U^{**} \in (P_2^*, qD_2)$ satisfying $V_U(P_U^{**}) = -\overline{V}$. When $\overline{V} \geq -V_U(qD_2)$, the regulator optimally chooses qD_2 to avoid saving insolvent banks.

□

Appendix B.9 Proof of Proposition 7

Proof. We consider a regulatory disclosure with no friction in communication. When s is realized, the regulator perfectly observes it and costlessly reveals it to the subsequent players. Knowing s, the creditors play bank run games, then the secondary market asset buyers bid for banks' assets upon runs happen. The combination $(\theta_s^*, x_s^*, P_s^*)$, $s = G$ or B, solves the system of equations.

$$P_s^* = \frac{\underline{\theta}_s + \theta_s^*}{2}, \quad x_s^* = \theta_s^* + (2q-1)\epsilon, \quad \theta_s^* = \frac{D_2 - D_1}{1 - qD_1/P_s^*}.$$

Follow Appendix B.2, we can solve analytically the unique combination $(x_s^*, \theta_s^*, P_s^*)$ for each $s \in \{G, B\}$. In particular, we have

$$\theta_s^* = \frac{\Psi_s + \sqrt{(\Psi_s)^2 - 8qD_1\underline{\theta}_s} - 2\underline{\theta}_s}{2}, \tag{B.48}$$

where $\Psi_s \equiv (D_2 - D_1) + 2qD_1 + \underline{\theta}_s$. Moreover, it can be verified that $1 > \omega_2^B(\theta)/\omega_2^G(\theta) > \omega_1^B(\theta)/\omega_1^B(\theta) > 0$, $\forall \theta \in [\theta^L, \theta^U]$. Follow Step 2 of Appendix B.3, we can show $\theta_G^* < \theta_1^* < \theta_2^* < \theta_B^*$. Since banks' cash flows are i.i.d. in a given s, the risk of systemic bank failures under the regulatory disclosure is

$$\mathrm{SYS}(\theta_G^*, \theta_B^*) = \alpha \cdot \left(\frac{\theta_G^* - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G}\right)^2 + (1-\alpha) \cdot \left(\frac{\theta_B^* - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B}\right)^2.$$

We focus on the regulator's liquidity support with the mutual commitments and zero loss-bearing capacity $\overline{V} = 0$. In this case, the equilibrium critical cash flow θ_A^* belongs to (θ_1^*, θ_2^*) and the risk of systemic bank failures is $\mathrm{SYS}(P_A^*) = \mathrm{SYS}(\theta_A^*)$. We establish a sufficient condition to guarantee $\mathrm{SYS}(\theta_A^*) < \mathrm{SYS}(\theta_G^*, \theta_B^*)$.

From (B.43), the regulator's break-even condition $V_A(P_A^*) = 0$ under the committed liquidity support can be expressed as

$$\alpha \cdot \left(\frac{\theta_A^* - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G}\right) \cdot \left(\frac{\theta_A^* + \underline{\theta}_G}{2} - P^*(\theta_A^*)\right) + (1-\alpha) \cdot \left(\frac{\theta_A^* - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B}\right) \cdot \left(\frac{\theta_A^* + \underline{\theta}_B}{2} - P^*(\theta_A^*)\right) = 0.$$

Recall that $P^*(\theta) = qD_1\theta/\left[\theta - (D_2 - D_1)\right]$ is the inverse function of $\theta^*(P)$. The regulator's break-even condition can be further expressed as $\omega_A^G(\theta_A^*)\cdot\pi(\theta_A^*|G) + \omega_A^B(\theta_A^*)\cdot\pi(\theta_A^*|B) = 0$, where

$$\omega_A^B(\theta_A^*) = \frac{(1-\alpha)\cdot\left(\frac{\theta_A^*-\underline{\theta}_B}{\overline{\theta}-\underline{\theta}_B}\right)}{\alpha\cdot\left(\frac{\theta_A^*-\underline{\theta}_G}{\overline{\theta}-\underline{\theta}_G}\right)+(1-\alpha)\cdot\left(\frac{\theta_A^*-\underline{\theta}_B}{\overline{\theta}-\underline{\theta}_B}\right)} = 1-\omega_A^G(\theta_A^*) \quad \text{and} \quad \pi(\theta_A^*|s) = \frac{\theta_A^*+\underline{\theta}_s}{2} - P^*(\theta_A^*).$$

We then consider a hypothetical case where the regulator offers a price $P^{ex} = P^*(\theta^{ex})$.

$$\alpha\cdot\left(\frac{\theta^{ex}+\underline{\theta}_G}{2} - P^*(\theta^{ex})\right) + (1-\alpha)\cdot\left(\frac{\theta^{ex}+\underline{\theta}_B}{2} - P^*(\theta^{ex})\right) = 0.$$

In other words, P^{ex} is the regulator's break-even price when she takes an ex-ante perspective about s. Note that θ^{ex} is the associated equilibrium critical cash flow. We have:

$$\frac{\omega_A^B(\theta)}{\omega_A^G(\theta)} = \left(\frac{1-\alpha}{\alpha}\cdot\frac{\overline{\theta}-\underline{\theta}_G}{\overline{\theta}-\underline{\theta}_B}\right)\cdot\frac{\theta-\underline{\theta}_B}{\theta-\underline{\theta}_G} < \frac{1-\alpha}{\alpha}, \qquad \forall\theta\in[\theta^L,\theta^U].$$

Following Step 2 of Appendix B.3, we have $\theta_A^* < \theta^{ex}$ and $P_A^* > P^{ex}$. Define an auxiliary function $\varphi(\theta) = 2qD_1\theta/[\theta-(D_2-D_1)] - \theta$. One can check that $\varphi' < 0$ and $\varphi'' > 0$. Note that θ^{ex} is the unique solution of $\varphi(\theta^{ex}) = \alpha\underline{\theta}_G + (1-\alpha)\underline{\theta}_B$ and θ_s^* is unique solution of $\varphi(\theta_s^*) = \underline{\theta}_s$ for each $s = G$ and B. By Jensen's inequality, we have $\varphi(\theta^{ex}) = \alpha\varphi(\theta_G^*) + (1-\alpha)\varphi(\theta_B^*) > \varphi\left(\alpha\theta_G^* + (1-\alpha)\theta_B^*\right)$. This means $\alpha\theta_G^* + (1-\alpha)\theta_B^* > \theta^{ex} > \theta_A^*$. Then, $\mathrm{SYS}(\theta_A^*) < \mathrm{SYS}(\theta_G^*,\theta_B^*)$ if and only if

$$\left(\theta_A^*-\underline{\theta}_G\right)^2 + \kappa\cdot\left(\theta_A^*-\underline{\theta}_B\right)^2 < \left(\theta_G^*-\underline{\theta}_G\right)^2 + \kappa\cdot\left(\theta_B^*-\underline{\theta}_B\right)^2.$$

Recall that $\kappa \equiv \frac{\alpha}{1-\alpha}\left(\frac{\overline{\theta}-\underline{\theta}_B}{\overline{\theta}-\underline{\theta}_G}\right)^2$. A sufficient condition to guarantee this inequality becomes

$$\left[\alpha\theta_G^* + (1-\alpha)\theta_B^* - \underline{\theta}_G\right]^2 + \kappa\cdot\left[\alpha\theta_G^* + (1-\alpha)\theta_B^* - \underline{\theta}_B\right]^2 < \left(\theta_G^*-\underline{\theta}_G\right)^2 + \kappa\cdot\left(\theta_B^*-\underline{\theta}_B\right)^2,$$

which can be reformulated into

$$\overline{\theta} > \underline{\theta}_G + \frac{\underline{\theta}_G - \underline{\theta}_B}{\sqrt{\frac{\alpha\theta_G^*+(2-\alpha)\theta_B^*-2\underline{\theta}_B}{(1+\alpha)\theta_G^*+(1-\alpha)\theta_B^*-2\underline{\theta}_G}} - 1} \equiv \overline{\theta}^c.$$

Recall the analytical form of θ_s^* in (B.48). The denominator is larger than zero since $\theta_G^* < \theta_B^*$ and $\underline{\theta}_G > \underline{\theta}_B$.

□